Mathematical Reviews

Edited by

R. P. Boas, Jr. **E. Hille** **H. Whitney**

J. V. Wehausen, *Executive Editor*

Vol. 14, No. 6 **June, 1953** **pp. 523–608**

TABLE OF CONTENTS

AUTHOR INDEX

(*Continued on cover 4*)

MATHEMATICAL REVIEWS

Published monthly, except August, by

THE AMERICAN MATHEMATICAL SOCIETY, Prince and Lemon Streets, Lancaster, Pennsylvania

Sponsored by

THE AMERICAN MATHEMATICAL SOCIETY
THE MATHEMATICAL ASSOCIATION OF AMERICA
THE INSTITUTE OF MATHEMATICAL STATISTICS
THE EDINBURGH MATHEMATICAL SOCIETY
SOCIÉTÉ MATHÉMATIQUE DE FRANCE
MATEMATISK FORENING I KØBENHAVN
HET WISKUNDIG GENOOTSCHAP TE AMSTERDAM
THE LONDON MATHEMATICAL SOCIETY
POLSKIE TOWARZYSTWO MATEMATYCZNE
UNIÓN MATEMÁTICA ARGENTINA
INDIAN MATHEMATICAL SOCIETY
UNIONE MATEMATICA ITALIANA

Editorial Office

MATHEMATICAL REVIEWS, 80 Waterman St., Providence 6, R. I.

Subscriptions: **Price $20 per year ($10 per year to members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions may be addressed to MATHEMATICAL REVIEWS, Lancaster, Pennsylvania, but should preferably be addressed to the American Mathematical Society, 80 Waterman St., Providence 6, R. I.**

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. Its preparation is also supported currently under a contract with the Office of Scientific Research, Headquarters, Air Research and Development Command, U. S. Air Force. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879. Accepted for mailing at special rate of postage provided for in paragraph (d-2), section 34.40, P. L. and R. of 1948, authorized November 9, 1940.

*Bac
à S
750
Thi
Greec
held
the I
metho
surve
in th
chapt
cosmo
innat
with
chapt
again
the S
clare
impo
the 5
rents
conne
signif
matio
ment
excur
logica
Anax
gorea
chapt
separ

Yulk
Mi
Is

Bard
du
(19
A
conn
comm
mati
cont
dyad
comm
the r

*Bo
H
ma
Jo
(L

Mathematical Reviews

Vol. 14, No. 6 JUNE, 1953 Pages 523-608

HISTORY

✶Baccou, Robert. Histoire de la science grecque de Thalès à Socrate. Editions Montaigne, Paris, 1951. 257 pp. 750 francs.

This book on the origin of scientific thinking in ancient Greece contains part of a course on the history of science, held by the author (who died in 1943 at the age of 36) at the Institut Catholique de Toulouse. An introduction on method in the history of science is followed by a concise survey of Egyptian and Babylonian mathematics, which are in the author's opinion devoid of scientific value. Three chapters are devoted to the school of Milete; the Ionian cosmological speculations are described as the result of an innate disinterested striving after truth, any connection with the development of technics being denied. In two chapters on Pythagoras and his school the author reacts against the searching criticisms to which the testimonies on the Sage of Samos were subjected not so long ago and declares himself convinced of his real existence and his high importance for the evolution of science. Then turning to the 5th century he deals with the main philosophical currents in this period, which are discussed primarily in their connection with mathematics, cosmology, and physics. The significance of the school of Elea for the history of mathematics is duly stressed; the controversies on Zeno's arguments in later times is made the subject of an historical excursus. After dealing with Empedocles (whose physiological and medical theories are given some attention), Anaxagoras and the Atomists, the author returns to Pythagoreanism as developed by Philolaos and Archytas. Five chapters on Hippocrates of Cos had to be reserved for a separate publication. *E. J. Dijksterhuis* (Oisterwijk).

Yuškevič, A. P. On the mathematics of the nations of Middle Asia in the 9th-15th centuries. Istor.-Mat. Issled. 4, 455–488 (1951). (Russian)

Barde, René. Recherches sur les origines arithmétiques du Yi-King. Arch. Internat. Hist. Sci. (N.S.) 5, 234–281 (1952).

A study of the numeral systems in ancient China in connection with the Chinese classic Yi-King and its various commentaries. To the reviewer the most important mathematical achievement in Yi-King lies in the fact that it contains one of the earliest elaborate treatments of the dyadic system. This was later mystified, with numerous commentaries. The paper also contains some accounts of the more mysterious aspects of the book. *S. Chern.*

✶Bohr, Harald. Presentation of a new edition of Zeuthen's History of mathematics. Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 195–200. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr. (Danish)

Bouligand, G. Attitudes de la pensée mathématique et histoire des sciences. Arch. Internat. Hist. Sci. (N.S.) 5, 230–233 (1952).

Neugebauer, O. Tamil astronomy. A study in the history of astronomy in India. Osiris 10, 252–276 (1952).

Viola, Tullio. Sulle origini della prospettiva. Il Filomate 1, no. 4, 9 pp. (1948).

The vanishing point of certain lines perpendicular to the picture plane and in the ground plane appears first in the "Annunciazione" of A. Lorenzetti (1344). It appears systematically in the paintings of Brunelleschi and his pupil Uccello; these painters have also other correct results. Some correct rules are described by Alberti and Della Francesca; the latter uses incidentally a distance point. The discovery of the vanishing point of a general set of parallel lines seems to have been impeded by the feeling of painters that parallel lines should not be represented by intersecting lines. The first appearance in a textbook of the vanishing point for lines parallel to the ground plane in arbitrary direction is in "De artificiali perspectiva" (1505) by Jean Pélerin, writing under the name of Viator; there is an indication that he also had knowledge of the vanishing point of an arbitrary system of parallel lines. The 1509 edition of this book exists in a phototypic reproduction [Paris, Tross, 1860]; another such reproduction has been made by W. M. Ivins, "On the rationalization of sight" [Metropolitan Museum of Art, New York, 1938]. The work of Pélerin considerably antidates that of Danti (1577, 1583) and Del Monte (1600), to whom the first publication of the general vanishing point has been previously ascribed. *D. J. Struik.*

✶Tambs Lyche, R. Isolated remarks on the history of mathematics in Norway. Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. xxi–xxxi. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr. (Norwegian)

Suškevič, A. K. Materials for the history of algebra in Russia in the 19th and the beginning of the 20th centuries. Istor.-Mat. Issled. 4, 237–451 (1951). (Russian)

✶Glagolev, A. A. Higher synthetic geometry in the works of N. D. Brašman, V. Ya. Cinger and K. A. Andreev. Nomografičeskiĭ sbornik [Nomographic collection], pp. 7–24. Izdat. Moskov. Gos. Univ., Moscow, 1951. (Russian)

Coolidge, J. L. The lengths of curves. Amer. Math. Monthly 60, 89–93 (1953).

Hille, Einar. Mathematics and mathematicians from Abel to Zermelo. Math. Mag. 26, 127–146 (1953).

✱The works of Archimedes including The Method. Great Books of the Western World, no. 11, pp. 397–592. Encyclopaedia Britannica, Inc., Chicago, London, Toronto, 1952.

This edition is from the Heath translation [Cambridge, 1897, 1912].

Clagett, Marshall. Archimedes in the middle ages: the De mensura circuli. Osiris 10, 587–618 (1952).

✱Bianchi, Luigi. Opere. Vol. I, Parte prima. A cura dell'Unione Matematica Italiana e col contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese della Casa Editrice Perrella, Roma, 1952. 615 pp. 5000 Lire.

This is the first part of volume 1 of a projected edition in ten volumes of Bianchi's mathematical work. It contains his papers on finite groups, algebraic equations and the theory of numbers with an introductory commentary by G. Ricci and his papers on infinite discontinuous groups and arithmetic forms with commentary by G. Sansone. In addition, there is a list of Bianchi's publications prepared by the late Enea Bortolotti, an obituary by G. Scorza, and an essay on his scientific work by G. Fubini and A. M. Bedarida.

✱Dürr, Karl. The propositional logic of Boethius. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1951. x+79 pp. 8 florins.

This book is concerned with two treatises of Boethius (475?–525?), the 'De Syllogismo Hypothetico' and parts of the commentary on Cicero's 'Topics'. The author first discusses Boethius' sources and his influence in the early scholastic period, especially upon Abélard. He then proceeds to a detailed analysis of the two treatises. This is done with care and in the light of the recent developments in mathematical logic. *R. M. Martin* (Philadelphia, Pa.).

Titchmarsh, E. C. Obituary: Harald Bohr. J. London Math. Soc. 28, 113–115 (1953).

Whitehead, J. H. C. Obituary: Elie Joseph Cartan, 1869–1951. Obit. Notices Roy. Soc. London 8, 71–95 (1 plate) (1952).

Hodge, W. V. D. Obituary: Elie Cartan. J. London Math. Soc. 28, 115–119 (1953).

Hodge, W. V. D. Obituary: Guido Castelnuovo. J. London Math. Soc. 28, 120–125 (1953).

Massey, H. S. W. Obituary: Leslie John Comrie, 1893–1950. Obit. Notices Roy. Soc. London 8, 97–107 (1 plate) (1952).

✱Načala Evklida. Knigi I-VI. [Euclid's Elements. Books I-VI.] Gosudarstv. Izdat. Tehn.-Teor. Lit. Moscow-Leningrad, 1950. 447 pp. 19 rubles.

✱Načala Evklida. Knigi VII-X. [Euclid's Elements. Books VII-X.] Gosudarstv. Izdat. Tehn.-Teor. Lit. Moscow-Leningrad, 1950. 511 pp. 21.20 rubles.

✱Načala Evklida. Knigi XI-XV. [Euclid's Elements. Books XI-XV.] Gosudarstv. Izdat. Tehn.-Teor. Lit. Moscow-Leningrad, 1950. 331 pp. 13.80 rubles.

The translation is by D. D. Morduhaĭ-Boltovskoĭ and was made from the Greek text of J. L. Heiberg's edition [Euclides opera omnia, vols. I–V, Teubner, Leipzig, 1883–1888]. The translation follows the Greek text closely, the translator having purposely refrained from introducing later terminology. An extensive commentary (comprising roughly half the bulk) by the translator traces the development of many of the ideas in later works, occasionally up to quite recent times: "One may state that my aim was to give firstly Euclid's Elements just as it was in the past, i.e. in its original form, and after that as it developed in the process of evolution of mathematical thought. . . ." There are occasional additional comments by M. Ya. Vygodskiĭ and I. N. Veselovskiĭ (who discusses, e.g., the authorship of Books XIV and XV).

✱The thirteen books of Euclid's Elements. Great Books of the Western World, no. 11. pp. vii–xi+1–396. Encyclopaedia Britannica, Inc., Chicago, London, Toronto, 1952.

This edition of the Elements is from the Heath translation [2nd ed., Cambridge, 1926].

Terracini, Alessandro. Obituary: Gino Fano. Boll. Un. Mat. Ital. (3) 7, 485–490 (1 plate) (1952).

✱Fourier, Jean Baptiste Joseph. Analytical theory of heat. Great Books of the Western World, no. 45, pp. 163–251. Encyclopaedia Britannica, Inc., Chicago, London, Toronto, 1952.

From the translation by Alexander Freeman [Cambridge Univ. Press, 1878].

Obituary: Yakov Il'ič Frenkel'. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. 23, 613–618 (1 plate) (1952). (Russian)

✱Huygens, Christiaan. Treatise on light. Great Books of the Western World, no. 34, pp. 545–619. Encyclopaedia Britannica, Inc., Chicago, London, Toronto, 1952.

From the translation by S. P. Thompson [Macmillan, London, 1912].

✱Huber, Kurt. Leibniz. Verlag von R. Oldenbourg, München, 1951. 451 pp. (1 plate).

Signorini, Antonio. Leonardo e la meccanica. Archimede 4, 221–227 (1952).

Signorini, Antonio. Leonardo e la meccanica. Ricerca Sci. 22, 2267–2274 (1952).

Yanovskaya, S. A. On the Weltanschauung of N. I. Lobačevskiĭ. Istor.-Mat. Issled. 4, 173–200 (1 plate) (1951). (Russian)

Laptev, B. L. The theory of parallel lines in the early works of N. I. Lobačevskiĭ. Istor.-Mat. Issled. 4, 201–229 (1951). (Russian)

Morozov, V. V. On the algebraic manuscripts of N. I. Lobačevskiĭ. Istor.-Mat. Issled. 4, 230–234 (1951). (Russian)

Pleijel, Åke. Johannes Malmquist in memoriam. Acta Math. 88, ix–xii (1952).

Rozenfel'd, B. A. On the mathematical works of Nasirèddin Tusi. Istor.-Mat. Issled. 4, 489–512 (1 plate) (1951). (Russian)

★Newton, Isaac. Optics. Great Books of the Western World, no. 34, pp. 373–544. Encyclopaedia Britannica, Inc., Chicago, London, Toronto, 1952.

Reprinted from the fourth edition [London, 1730].

★Newton, Isaac. Mathematical principles of natural philosophy. Great Books of the Western World, no. 34, pp. vii–xi+1–372. Encyclopaedia Britannica, Inc., Chicago, London, Toronto, 1952.

Reprinted from the Motte translation as revised by Cajori [Univ. of California Press, Berkeley, 1934].

★Nicomachus of Gerasa. Introduction to arithmetic. Great Books of the Western World, no. 11, pp. 805–848. Encyclopaedia Britannica, Inc., Chicago, London, Toronto, 1952.

Reprinted from the translation by D'Ooge, Robbins and Karpinski [Macmillan, New York, 1926].

Mushelišvili, N. Giorgi Nikoladze. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 15, 1–17 (2 plates) (1947). (Georgian)

According to a footnote, this is reprinted with a few alterations from a biographical essay by Mushelišvili in the beginning of Nikoladze's posthumously published "Elements of differential geometry" (in Georgian) [Sahelgami, 1934].

Remez, E. Ya. On the mathematical manuscripts of Academician M. V. Ostrogradskiĭ. Istor.-Mat. Issled. 4, 9–98 (1 plate) (1951). (Russian)

Gnedenko, B. V. On M. V. Ostrogradskiĭ's works on the theory of probability. Istor.-Mat. Issled. 4, 99–123 (1951). (Russian)

Maron, I. A. The general pedagogical views of M. V. Ostrogradskiĭ. Istor.-Mat. Issled. 4, 124–159 (1951). (Russian)

Depman, I. Ya. Supplementary information on the pedagogical activity of M. V. Ostrogradskiĭ. Istor.-Mat. Issled. 4, 160–170 (1951). (Russian)

Born, Max. Obituary: Arnold Johannes Wilhelm Sommerfeld, 1868–1951. Obit. Notices Roy. Soc. London 8, 275–296 (1 plate) (1952).

Whittaker, E. T. Obituary: Arnold Johannes Wilhelm Sommerfeld. J. London Math. Soc. 28, 125–128 (1953).

Conte, Luigi. Vincenzo Viviani e l'invenzione di due medie proporzionali. Period. Mat. (4) 30, 185–193 (1952).

Yates, F. Obituary: George Udny Yule, 1871–1951. Obit. Notices Roy. Soc. London 8, 309–323 (1 plate) (1952).

FOUNDATIONS

★Kleene, Stephen Cole. Introduction to metamathematics. D. Van Nostrand Co., Inc., New York, N. Y., 1952. x+550 pp. $8.75.

"The aim of this book is to provide a connected introduction to the subjects of mathematical logic and recursive functions in particular, and to the newer foundational investigations in general." Dieses Ziel hat der Verf. auf eine vorbildliche Art erreicht. Neben dem Standardwerk von Hilbert-Bernays liegt damit jetzt eine Einführung in die mathematische Grundlagenforschung vor, die zum Selbststudium und als "textbook" für Vorlesungen uneingeschränkt empfohlen werden kann. Die Übersichtlichkeit des Werkes wird erhöht durch die Beschränkung auf die elementare Prädikatenlogik und Arithmetik. Innerhalb dieses Rahmens dringt das Buch überall bis zu dem neuesten Untersuchungen vor, insbesondere bzgl. der rekursiven Funktionen. Teil I enthält eine Einführung in die Problematik der Grundlagen der Mathematik. Nach den Grundbegriffen der Cantor'schen Mengenlehre, der Arithmetik und der Analysis werden die Paradoxien des naiven Mengenbegriffs und die Ansätze zu ihrer Auflösung (Logizismus, Intuitionismus, Beweistheorie) besprochen. Das in den nächsten Teilen durchgeführte Programm der Metamathematik wird ausführlich erläutert. Verf. rechnet dabei—wie es ja dem eigentlichen Sinn allein entspricht—zur Metamathematik nur die mit finiten Mitteln beweisbaren Theoreme über Formalismen. Die im Buch enthaltenen nicht-finiten Resultate sind stets extra gekennzeichnet.

Teil II diskutiert Aussagen- und Prädikatenkalkül und die elementare Arithmetik (einschl. Addition und Multiplikation). Der Aussagenkalkül enthält das Axiom (1) $\neg\neg A \supset A$ und geht bei Ersetzung von (1) durch (2) $\neg A \supset (A \supset B)$ in den intuitionistischen Kalkül über. Die Theoreme, die nur für die klassischen Formalismen gelten, sind wiederum gekennzeichnet. Die nur für die intuitionistischen Formalismen wichtigen Theoreme sind stets ausführlich behandelt. Die Untersuchung führt bis zum Gödel'schen Unvollständigkeitstheorem, auch in der Rosser'schen Form. Für die hieraus folgende Unbeweisbarkeit (im Formalismus) der Widerspruchsfreiheit wird—um das Verständnis nicht zu erschweren—auf die Durchführung der Arithmetisierung des metamathematischen Beweises verzichtet. In Teil III sind die rekursiven Funktionen behandelt. Verf. gibt hier—unter Heranziehung seiner eigenen Ergebnisse—eine meisterhafte Darstellung des Gebietes mit vielen Beweisvereinfachungen (primitiv-rekursive Funktionen, arithmetische Prädikate, Unvollständigkeit allgemeiner Formalismen, partielle rekursive Funktionen). Von den Definitionsmöglichkeiten der Rekursivität wird auch die Berechenbarkeit nach Turing ausführlich besprochen. Es wird hier wie im ganzen Buch grösster Wert darauf gelegt, nicht nur die Begriffe zu definieren, sondern auch die Definitionen zu rechtfertigen.

Teil IV nimmt das Thema von Teil II wieder auf. Zunächst werden das Gödel'sche Vollständigkeitstheorem (auch in seiner finiten Form), die Skolem'schen Sätze über die Axiomatisierungen der Mengenlehre und der Arithmetik und das Church'sche Unentscheidbarkeitstheorem behandelt, anschliessend der Gentzen'sche Hauptsatz und die Widerspruchsfreiheitsbeweise, soweit sie ohne die transfinite Induktion bis ϵ_0 geführt werden können. (Der volle Widerspruchsfreiheitsbeweis ist skizziert.) Das aus dem Hauptsatz folgende Entscheidungsverfahren für den intuitionistischen Aussagenkalkül ist dargestellt und einige Unableitbarkeitsbeweise für den intuitionistischen Prädikatenkalkül. Die Reduktion der klassischen Formalismen auf die intuitionistischen, sowie schliesslich die interessante Interpretation der intuitionistischen Arithmetik mit Hilfe der rekursiven Realisierbarkeit des Verf. findet sich hier.

Es ist überall gelungen, die schwierige Materie durch viele Beispiele zu klären. *P. Lorenzen* (Bonn).

***Rosser, J. Barkley, and Turquette, Atwell R. Many-valued logics.** Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1951. vii+124 pp. $3.25.

In this book, the previous work of the authors in many-valued logic [J. Symbolic Logic 10, 61–82 (1945); 13, 177–192 (1948); 14, 219–225 (1950); 16, 22–34 (1951); these Rev. 7, 185; 10, 420; 11, 709; 12, 791] is assembled with revisions and amplification. The introductory chapter, in the form of a dialogue between a Mr. Turquer and a Mr. Rossette, is concerned with questions of interpretation and applicability of many-valued systems.

"Truth-value stipulations" of many-valued statement calculi are considered in the second chapter. Such a stipulation is characterized by a set of statements and statement functions, M truth values and S designated values where $1 \leqq S < M$. To each statement function there corresponds a truth-value function. An "acceptable" statement is one whose corresponding truth value is always designated. A calculus is called "functionally complete" if "the totality of constructable functions comprises all possible truth functions of the truth values $1, \cdots, M$". Attention is focused on many-valued analogues of the familiar two-valued operators $\supset$, $\sim$, $\vee$, &. Where such an analogy holds—e.g., the value of $\supset(p, q)$ is undesignated if and only if the value of p is designated and the value of q is undesignated; the value of $\sim(p)$ is designated if and only if p is undesignated—the corresponding many-valued operator is said to satisfy standard conditions. It is then demonstrated that, although the Łukasiewicz-Tarski C and N [C. R. Soc. Sci. Varsovie 23, 30–50 (1930)] are inadequate as generalizations for the two-valued $\supset$ and $\sim$, analogues to the familiar two-valued operators can be defined in a system based on C and N. This is achieved by first defining in terms of C and N a set of functions $J_k(P)$ $(1 \leqq k \leqq M)$ whose corresponding truth functions take the value 1 when the value associated with P is k, and M when the value associated with P is not k. Such a system is however functionally incomplete. If the Słupecki T [ibid. 29, 9–11 (1936)] is also taken as basic, then the system is functionally complete.

The third chapter is concerned with axiomatizations of many-valued statement calculi. Two "axiomatic stipulations" are proposed, an A-stipulation and an alternative B-stipulation. These involve two uninterpreted functions $P \supset Q$ and $J_k(P)$ $(1 \leqq k \leqq M)$, axiom schemes and rules. Equivalence of an axiomatic stipulation to a given truth-value stipulation is defined by deductive completeness and plausibility (consistency) with respect to the given truth-value stipulation. The principal results of this chapter relate to such equivalences. The A-stipulation is shown to be deductively complete and, where standard conditions are fulfilled, also plausible with respect to a given truth-value stipulation. Examples are given of particular many-valued statement calculi equivalent to given truth-value stipulations for specific determinations of S, M, the basic and defined functions. Some of these calculi are functionally incomplete.

The alternative B-stipulation is shown to be at least as strong as the A-stipulation; depending on the definitions of $P \supset Q$ and $J_k(P)$, it is stronger or equivalent to the A-stipulation.

In the fourth chapter, the notion of a truth-value stipulation is extended to many-valued systems with individual variables, predicates, and quantifiers which permit simultaneous binding of several variables. The truth function associated with a statement function involving quantifiers is obtained by the device of partial normal forms. Standard conditions are defined for a particular many-valued quantifier (X) which is analogous to the familiar universal quantifier—the truth-value associated with $(X)P$ is designated if and only if the truth-value associated with P is designated.

Axiomatizations of many-valued predicate calculi are considered in the fifth chapter. The A-stipulation is extended to include the many-valued function $\sim P$, the quantifier (X), additional axiom schemes, and a generalization rule. On the assumption that standard conditions are fulfilled, the system is shown to be plausible and deductively complete with respect to a given truth-value stipulation. The completeness proof is a generalization of Henkin's proof for the two-valued predicate calculus [J. Symbolic Logic 14, 159–166 (1949); these Rev. 11, 487]. A generalized Skolem-Löwenheim theorem follows directly as a corollary. The authors then examine some particular many-valued predicate calculi for specific choices of M, S, basic functions and quantifiers. Attention is confined to systems which are demonstrably equivalent to given truth-value stipulations. Here again, functional completeness is not required. An illustration from modal logic indicates that modal predicate calculi constructed in this way would be assured of deductive completeness as contrasted for example with the method of Barcan [ibid. 11, 1–16 (1946); these Rev. 8, 125].

The book concludes with a list of open problems in the theory of many-valued logic and its applications.

R. Barcan Marcus (Evanston, Ill.).

Skolem, Th. On the proofs of independence of the axioms of the classical sentential calculus. Norske Vid. Selsk. Forh., Trondheim 24 (1951), 20–25 (1952).

This paper gives some simplifications of the Hilbert and Ackermann [Grundzüge der theoretischen Logik, 2nd ed., Springer, Berlin, 1938] proofs of the independence of the axioms for the classical sentential calculus.

R. M. Martin (Philadelphia, Pa.).

Dreben, Burton. On the completeness of quantification theory. Proc. Nat. Acad. Sci. U. S. A. 38, 1047–1052 (1952).

Für den Herbrand'schen Satz wird in Verbindung mit dem Gödel'schen Vollständigkeitssatz ein einfacher Beweis gegeben. Zu jeder pränexen Formel ϕ des Prädikatenkalküls wird wie bei Hilbert-Bernays eine Folge aussagenlogischer Formeln $\psi_1, \psi_2, \cdots$ definiert. Ist die Disjunktion $\psi_1 \vee \psi_2 \vee \cdots \vee \psi_n$ für ein n tautologisch, dann lässt sich ϕ hieraus allein durch Einführungsregeln für die Quantoren ableiten. Der Beweis des Gödel'schen Satzes liefert, dass $-\phi$ erfüllbar ist, wenn $\psi_1 \vee \psi_2 \vee \cdots \vee \psi_n$ für kein n tautologisch ist. Auf diesem nicht-finiten Wege folgt sofort für jedes ableitbare ϕ die Existenz eines n, so dass $\psi_1 \vee \psi_2 \vee \cdots \vee \psi_n$ tautologisch ist. *P. Lorenzen* (Bonn).

Myhill, John. A finitary metalanguage for extended basic logic. J. Symbolic Logic 17, 164–178 (1952).

This paper is concerned with the problem of formalizing the syntactical meta-language of Fitch's system K' [same J. 7, 105–114 (1942); 9, 57–62 (1944); 13, 95–106 (1948); 14, 9–15 (1949); 15, 17–24 (1950); these Rev. 4, 125; 6, 197; 9, 559; 10, 669; 12, 2; 13, 4]. Fitch uses an unformalized meta-language of great power, containing presumably a functional calculus of second order. Myhill shows how, using

the syntactical methods of Chwistek, Hetper, and Quine, the underlying logic can be kept at first order. Arithmetic and the theory of rationals may be built up using familiar methods. But also, using an adaptation of Fitch's proof of a syntactical analogue of the least upper bound theorem, Myhill is able to prove this theorem within the restricted, first-order meta-language. Also syntactical analogues of the Heine-Borel and Bolzano-Weierstrass theorems are provable as well as a syntactical form of an axiom of choice. But in Myhill's meta-language the consistency of K' apparently cannot be proved; it must rather be assumed by way of an axiom. Thus there seems little reason to prefer this meta-language to the more powerful (albeit unformalized) one of Fitch, wherein the consistency of K' is provable. The problem of knowing precisely what is assumed in Fitch's consistency proof is still left open. *R. M. Martin.*

Myhill, John. A derivation of number theory from ancestral theory. J. Symbolic Logic 17, 192–197 (1952).

Using only an operation of forming the ordered pair together with the notion of the ancestral of any dyadic relation thereby definable, it is shown in this paper that elementary arithmetic can be formalized. The treatment of the ancestral is similar to that of the reviewer [same J. **8**, 1–23 (1943); these Rev. **4**, 182], although the basic logic in Myhill's treatment is more powerful, being a non-simple applied functional calculus of first order. The only axioms needed here are two for the ordered pair and three for the ancestral. *R. M. Martin* (Philadelphia, Pa.).

Curry, Haskell B. The permutability of rules in the classical inferential calculus. J. Symbolic Logic 17, 245–248 (1952).

The author gives a general condition for the permutability of two consecutive applications of rules of the Gentzen calculus. As these conditions are fulfilled in the classical calculus, the result can be used to derive Gentzen's Hauptsatz in its strengthened form [O. Ketonen, Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **23** (1944); these Rev. 8, 125]. *A. Heyting* (Amsterdam).

Curry, Haskell B. The elimination theorem when modality is present. J. Symbolic Logic 17, 249–265 (1952).

In order to extend Gentzen's elimination theorem to the theory of modalities contained in Chapter V of his "A theory of formal deducibility" [Univ. of Notre Dame, Notre Dame, Ind., 1950; these Rev. **11**, 487], the author gives a new proof for this theorem, which applies to formal systems in which the rules satisfy certain general conditions. Though the rules for necessity do not satisfy these conditions, the proof can easily be adapted to them. According to the author's definition of possibility [loc. cit., chap. V, §4], $\Diamond A$ holds in the "inner system" S if and only if A holds in the "outer system" S'. The relation of entailment in S is designated by $\Vdash$, that in S' by $\Vvdash$. Let S' be obtained by adding to S certain rules of the form $A_1, \cdots, A_n \vdash B$. A prosequence $\mathfrak{M}$ represents S' in S if for each of these rules we have $\mathfrak{M}, A_1, \cdots, A_n \Vdash B$. It is assumed that $\mathfrak{M}$ consists of a single proposition M. Then the system LXZ is obtained from LX by adjoining $\Diamond$ and the following rules: If $\mathfrak{Z} \leq \mathfrak{Y}$, then

$$\frac{\mathfrak{X} \Vdash M, \mathfrak{Z} \quad \mathfrak{X}, A \Vdash \mathfrak{Y}}{\mathfrak{X}, \Diamond A \Vdash \mathfrak{Y}}; \quad \frac{\mathfrak{X}, M \Vdash A, \mathfrak{Z}}{\mathfrak{X} \Vdash \Diamond A, \mathfrak{Z}}.$$

Every proposition P of LXZ can be translated into a proposition P^* of LX, essentially by substituting $M \supset A$ for every occurrence of $\Diamond A$. Then P is valid in LXZ if and only if P^* is valid in LX. This proves the elimination theorem for LXZ, provided it is true for LX. *A. Heyting.*

Ridder, J. Über modale Aussagenlogiken und ihren Zusammenhang mit Strukturen. I. Nederl. Akad. Wetensch. Proc. Ser. A. 55 = Indagationes Math. 14, 213–223 (1952).

The author presents several systems of modal logic obtained by addition of axioms and rules of inference for possibility and necessity operators to the systems of logic discussed in his earlier papers [same Proc. **53**, 327–336, 446–455 (1950); these Rev. **11**, 636; **12**, 71]. The present work discusses subsystems of the modal systems presented in an earlier paper [ibid. **54**, 169–177 (1951); these Rev. **13**, 310]. He establishes inclusion relations among the various logics and discusses the relation of these systems to those of Lewis [C. I. Lewis and C. H. Langford, Symbolic logic, Century, New York, 1932]. *D. Nelson.*

***Poirier, René. Logique et modalité du point de vue organique et physique.** Actualités Sci. Ind., no. 1163. Hermann et Cie., Paris, 1952. 113 pp. 600 francs.

This is a philosophical discussion about various interpretations of modalities. The author examines more closely the "organic" (psychological) interpretation which is intended to represent the experimentally discovered methods of actual reasoning. He introduces as fundamental concepts $a.b$ (a and b), $\supset a$ (a is provable), $\subset a$ (a is refutable, excluded), $\bar{a}$ (a is factually unprovable), $a \supset b$ (a implies b), $a \subset b$ (a excludes b). Two forms of alternation are introduced as follows: $a \smile b :=_{\mathrm{D}}: \subset a . \supset . \supset b : \subset b . \supset . \supset a$; $a\, \omega\, b :=_{\mathrm{D}}: \bar{a} . \supset . \supset b : \bar{b} . \supset . \supset a$. Several formulas are examined as to their validity, e.g., the principle of excluded middle is valid in the form $\bar{a} . \smile . \supset a$, but not in the form $\subset a . \smile . \supset a$, and in neither of these forms with ω instead of $\smile$. In a note, quantification is examined from the same point of view. A completely formalized theory is promised for another publication. *A. Heyting* (Amsterdam).

***Goodstein, R. L. The foundations of mathematics.** University College, Leicester, 1951. 27 pp. 1 shilling.

The author's inaugural lecture sketches the historical development of the constructivist movement in mathematics and presents in outline his own views on foundational problems. These views are stated in greater detail in his Constructive formalism [University College, Leicester, 1951; these Rev. **14**, 123]. Of particular interest is the author's objection to the admission of the entire class of general recursive functions into constructive mathematics; he does accept as satisfactorily defined all transfinite recursions of ordinal less than ϵ. The lecture also takes exception to Church's [Analysis **10**, 97–99 (1950)] as an argument for the indispensability of propositions as entities distinct from statements. It should be mentioned in connection with the author's remarks on page 9 concerning the Russell paradox and the principle of excluded middle that the paradox may be derived without application of that principle. Brouwer's name has been consistently misprinted as "Bronwer". *D. Nelson* (Washington, D. C.).

Schütte, Kurt. Eine Bemerkung über quasirekursive Funktionen. Arch. Math. Logik Grundlagenforsch. 1, 63–64 (1951).

It was shown by Kleene [Bull. Amer. Math. Soc. **42**, 544–546 (1936)] that the class of general recursive functions

can be generated by substitution from addition, multiplication, the Kronecker δ-function, and the μ-operator applied to certain relations of the form $f(x_1, \cdots, x_n)=0$, where f has been previously defined. The author shows that the Kronecker δ-function may be eliminated if the constant function 1 is allowed and the μ-operator is applied to certain relations of the form $f(x_1, \cdots, x_n)=g(x_1, \cdots, x_n)$, where f and g have been previously defined. *D. Nelson.*

Kalmár, László. Another proof of the Markov-Post theorem. Acta Math. Acad. Sci. Hungar. **3**, 1–27 (1952). (Russian summary)

The theorem in question is that which states the recursive unsolvability of the word-problem for associative systems. The earliest proofs are by E. L. Post [J. Symbolic Logic **12**, 1–11 (1947); these Rev. **8**, 558] and (in outline) A. Markov [C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 583–586 (1947); these Rev. **8**, 558]. The present proof relies on Kleene's example of a numerical recursive function $R(x, y)$, such that no algorithm can be found for deciding whether, given $k\geqq 0$, there exists $y\geqq 0$ such that $R(k, y)=0$ [Math. Ann. **112**, 727–742 (1936)].

The central idea of the proof is to modify the formalism in which the defining "equations" of R are expressed, so as to dispense with brackets, in the manner of Łukasiewicz. A number of supplementary equations are introduced which do the work of substitution rules. If the expressions on either side of equations are now regarded as words in an associative system, having the defining equations of R and the supplementary equations as defining relations, it is shewn that a solution of the word-problem for this system would lead to a decision method for Kleene's problem and hence is impossible.

In contrast to much recent work in this field, proofs are given in full detail. *M. H. A. Newman* (Manchester).

Kalmár, László. New results connected with the foundations of mathematics. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei **2** (1951), 89–103; discussion, 104–112 (1952). (Hungarian)

An expository lecture on recent contributions to the foundations of mathematics by Hungarians (i.e., by Kalmár, Peter, and their students). The lecturer and the contributors to the subsequent discussion (Rényi, Alexits, Aczél) are much concerned with the relation of their subject to dialectical materialism; according to Kalmár, Church's theorem is in perfect harmony with that philosophy.

P. R. Halmos (Chicago, Ill.).

Brouwer, L. E. J. Historical background, principles and methods of intuitionism. South African J. Sci. **49**, 139–146 (1952).

Following remarks on the history of mathematics and the psychological foundations of intuitionism, the author presents examples of intuitionistic reasoning. He describes properties of natural numbers which lead the intuitionist not to accept the principle of excluded middle. He defines points of the intuitionistic unit square by nested sequences of binary squares. Each point is coincident with a standard point defined by a nested sequence with nth element of edge $2^{-(4n+1)}$. The standard points are a special case of what the author calls "a fan." A basic theorem for fans is stated. The author also observes an intuitionistic objection to the concept of measure for the classical Cartesian plane and notes that this objection does not arise for the intuitionistic plane. *D. Nelson* (Washington, D. C.).

Rostand, François. La notion de scrupule dans la psychologie des mathématiques. Rev. Gén. Sci. Pures Appl. **59**, 325–336 (1952).

Hönl, Helmut. Über das Mach'sche Prinzip. Z. Naturforschung **8a**, 2–6 (1953).

ALGEBRA

Popadić, Milan S. On the number of chains in a kind of ordered finite sets. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **4**, no. 4, 10 pp. (1951). (Serbo-Croatian. English summary)

The number of chains of length k (i.e., partially ordered sets with k elements in which any two elements are comparable) among all subsets of a set of n elements, partially ordered by set inclusion, is shown to be given by

$$\Delta^{k-1}2^n=\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}(k+1-j)^n$$

J. Riordan (New York, N. Y.).

Leavitt, W. G. Canonical forms for mappings of vector spaces. Amer. Math. Monthly **60**, 75–79 (1953).

A condensed derivation of known facts about the rational and the Jordan canonical forms of matrices.

P. R. Halmos (Chicago, Ill.).

Cherubino, Salvatore. Sopra certe famiglie di matrici. Ann. Scuola Norm. Super. Pisa (3) **6**, 69–73 (1952).

Capital letters denoting matrices and accents ordinary differentiation, this paper proves: (i) if $Y'=YP$, $\det Y\neq 0$, $PA=-AP^T$, $A'=0$, then $(YAY^T)'=0$; and (ii) if $YAY^T=B$, $A'=B'=0$, $\det(ABY)\neq 0$, $P=Y^{-1}Y'$, $Q=Y'Y^{-1}$, then $PA=-AP^T$, $QB=-BQ^T$. Index C denoting the complex conjugate, for $A=A^C=A^T=A^{-1}$ these results were given by MacDuffee [Proc. Amer. Math. Soc. **2**, 794–797 (1951); these Rev. **13**, 330]. The present author remarks that the results apply to matrices which are holomorphic functions of a matrix variable. *J. M. Thomas* (Durham, N. C.).

Lidskiĭ, V. B. The proper values of the sum and product of symmetric matrices. Translated by C. D. Benster. U. S. Department of Commerce, National Bureau of Standards, Washington, D. C. N.B.S. Rep. **2248**. 8 pp. (1953).

Translated from Doklady Akad. Nauk SSSR (N.S.) **75**, 769–772 (1950); these Rev. **12**, 581.

Leum, Mark, and Smiley, M. F. A matric proof of the fundamental theorem of algebra for real quaternions. Amer. Math. Monthly **60**, 99–100 (1953).

The fact that every one-sided polynomial equation in quaternions has a root follows from the results of L. Wolf [Bull. Amer. Math. Soc. **42**, 737–743 (1936)].

J. L. Brenner (Pullman, Wash.).

Abstract Algebra

Dubreil-Jacotin, L., et Croisot, R. Equivalences régulières dans un ensemble ordonné. Bull. Soc. Math. France **80**, 11–35 (1952).

The authors study various concepts of regularity of an equivalence with respect to an order relation. Let E be an ordered set. An equivalence $\mathfrak{R}$ on E is called regular (r) in case $a_i \equiv a_i'$, $a_i' \leqq a_{i+1}$ ($i=1, \cdots, n$), where $a_{n+1}=a_1$, always imply $a_1 \equiv a_2 \equiv \cdots \equiv a_n$. $\mathfrak{R}$ is called normally regular (nr) if (1) $a' \leqq b$, $b \equiv b'$, $b' \leqq c$ imply the existence of $a \equiv a'$ and of $c' \equiv c$ with $a \leqq c'$; and (2) $a' \leqq b$, $b \equiv b'$, $b' \leqq a$, $a \equiv a'$ imply $a \equiv b$. Upper strong regularity (usr) is defined by (S) $a' \equiv a$, $a' \leqq b$ imply the existence of $b' \equiv b$ with $a \leqq b'$; and (C) $a \leqq b \leqq a'$, $a \equiv a'$ imply $a \equiv b$. Lower strong regularity (lsr) is defined dually. Finally, total regularity (tr) means that $a \equiv a'$, $b \equiv b'$, $a \not\equiv b$, $a<b$ imply $a'<b'$. These concepts are related as follows: tr→usr, lsr; usr→nr, lsr→nr; nr→r. Various facts on the lattices of all equivalences regular (in one of these senses) with respect to a given order relation are proved. *L. Nachbin* (São José dos Campos).

Carruth, Philip W. Products of ordered systems. Proc. Amer. Math. Soc. **3**, 983–987 (1952).

Using the same definition of product of ordered systems as in his earlier paper [same Proc. **2**, 896–900 (1951); these Rev. **13**, 425] the writer gives conditions on the factors and index system for desirable properties, such as chain conditions or complementation, in the product. *M. M. Day.*

Takeuchi, Kensuke. On free modular lattices. Jap. J. Math. **21** (1951), 53–65 (1952).

The author presents results which are likely to be useful in attacking the word problem in free modular lattices, and some related results. In particular, he finds that if one can proceed from a to b ($a \leqq b$) by successive applications of lattice and modular laws, then there exists such a succession in which each application of the modular law is in the same direction, i.e., proceeds from $\cdots f \cup (g \cap h) \cdots$ to $\cdots (f \cup g) \cap h \cdots$ [according to the author's definitions, all $\leqq$ relations are required to hold in the free lattice rather than merely in the free modular lattice, but it appears that this requirement might be relaxed]. There are some results on infinite chains in lattices, including a proof of the existence of an order-convergent infinite chain in free lattices with infinitely many generators, answering for this case a problem posed by the reviewer [Ann. of Math. (2) **43**, 104–115 (1942); these Rev. **3**, 261]. *P. M. Whitman.*

Iseki, Kiyoshi. On a theorem of Stone-Samuel. Bull. Calcutta Math. Soc. **43**, 175–177 (1951).

A short proof of known results about conditions on a semilattice L equivalent to "L is a Boolean algebra".
P. M. Whitman (Silver Spring, Md.).

Pickert, Günter. Bemerkungen über Galois-Verbindungen. Arch. Math. **3**, 285–289 (1952).

The author examines the concept of Galois connections between partially ordered sets introduced by the reviewer. They are usually defined as a double correspondence satisfying the conditions

$$(*) \quad x_1 \leqq x_2 \rightarrow x_1^* \geqq x_2^*, \quad y_1 \leqq y_2 \rightarrow y_1' \geqq y_2', \quad x \leqq x^{*\prime}, \quad y \leqq x'^*.$$

It is shown that the definition may be reduced to a single correspondence satisfying (*) and a second condition on the existence of unions and their images. In particular, the Galois connections between two complete lattices A and B are given by those mappings of A into B which take upper bounds into lower bounds. The results are applied to symmetric binary relations and their associated closure operations. *O. Ore* (New Haven, Conn.).

Aubert, Karl Egil. Sur une généralisation de la théorie des idéaux dans un anneau commutatif sans condition de chaîne. C. R. Acad. Sci. Paris **236**, 31–33 (1953).

Let L be a Boolean algebra with a multiplication that is associative, commutative, and distributive relative to the union operation. Also let L have an operation of subtraction that is distributive relative to the union operation and is such that $a(b-c) \subseteq ab-ac$. If a is an ideal of L, then the union of all $x \in L$ such that $x^n \subseteq a$ for some integer $n \geqq 1$ is called the radical of a. In such a system as L, the author announces that he can develop the theory of Krull [Math. Ann. **101**, 729–744 (1929)] on ideals in a commutative ring without chain conditions. For example, he states that the radical of an ideal in L is the intersection of all prime ideals containing the ideal, and the radical of a weakly primary ideal in L is prime. *R. E. Johnson.*

Hattori, Akira. On invariant subrings. Jap. J. Math. **21** (1951), 121–129 (1952).

L'auteur généralise le théorème de H. Cartan-R. Brauer-L. K. Hua sur les sous-corps d'un corps non commutatif invariants par tous les automorphismes intérieurs, de la façon suivante: si A est un anneau simple satisfaisant à la condition minimale, tout sous-anneau B de A invariant (globalement) par tout automorphisme intérieur de A et satisfaisant à la condition minimale est nécessairement égal à A ou contenu dans le centre de A, à l'exception du cas de l'anneau A des matrices d'ordre 2 sur le corps à 2 éléments. La méthode (généralisation de celle utilisée par R. Brauer) consiste à utiliser les éléments $a \in A$ tels que a et $a+\alpha$ soient inversibles, α étant un élément convenable $\neq 0$ du centre de A; on montre alors successivement que B est semi-simple, puis que son centre est contenu dans le centre Z de A (ce qui entraîne que B est simple), et enfin, en considérant un système d'unités matricielles dans B, on prouve que le centralisateur C de B dans A est contenu dans Z. L'auteur montre comment son théorème s'étend aux anneaux semi-simples et aux anneaux primaires. Il montre enfin que si A est de rang fini sur son centre Z, il n'existe pas, dans le groupe quotient A^*/Z^* de couples de sous-groupes distingués M, N non réduits à l'identité tels que tout élément de M permute avec tout élément de N, à moins que A ne soit l'anneau des matrices d'ordre 2 sur un corps à 2 ou 3 éléments. *J. Dieudonné* (Ann Arbor, Mich.).

Nagata, Masayoshi. Corrections to my paper "On the structure of complete local rings." Nagoya Math. J. **5**, 145–147 (1953).

See same J. **1**, 63–70 (1950); these Rev. **13**, 7.

Nagata, Masayoshi. An ideal-theoretic observation on valuations. Sûgaku **4**, 76–80 (1952). (Japanese)

A simpler approach to Krull's theory of general valuations is given, and Hensel's lemma is treated without making use of completion by the methods in the author's paper reviewed below. *T. Nakayama* (Nagoya).

Nagata, Masayoshi. On the theory of Henselian rings. Nagoya Math. J. **5**, 45–57 (1953).

Suppose that $\mathfrak{o}$ is an integrally closed ring with the quotient field K. The author investigates the generalization of

Hilbert's theory of the decomposition group, etc., for prime ideals in the integral closure of $\mathfrak{o}$ within the maximal separable closure $\bar{K}$. The work, which is essentially of ideal-theoretic nature, is applied to a discussion of variants of Hensel's lemma for valuation rings. A ring $\mathfrak{o}$ is called locally Henselian at a prime ideal $\mathfrak{p}$ if the following holds: If a monic polynomial $f(x)$ with coefficients in $\mathfrak{o}$ factors modulo $\mathfrak{p}$ into a product of monic polynomials $g_0(x)$ and $h_0(x)$ and if their resultant is not in $\mathfrak{p}$, then $f(x)$ factors into a product of monic polynomials $g(x)$ and $h(x)$ such that $g(x)\equiv g_0(x)$ and $h(x)\equiv h_0(x)$ modulo $\mathfrak{p}$. Furthermore, $\mathfrak{o}$ is called Henselian if it is quasi-local and locally Henselian for its maximal prime ideal. The theory of the decomposition group, as indicated above, is then used to provide essentially unique algebraic extension rings of a given ring $\mathfrak{o}$, which are locally Henselian or Henselian, where the latter is the quotient ring of the locally Henselian extension ring with respect to the extenting prime ideal. These rings have the customary ideal-theoretic properties of a decomposition field or a completion for a valuation ring. Next the author applies his results to valuation rings, but pushes the usual arguments referring to value groups into the background, and proves the general form of Hensel's lemma [involving resultants in a form similar to that given by Rychlik in J. Reine Angew. Math. **153**, 94–107 (1923)]. Finally, an independence theorem for finite sets of valuation rings (no ring is a subring of any other of the given rings) is proved. *O. F. G. Schilling.*

Dürbaum, Hansjürgen. Über die Ganzheitsbereiche bewerteter Körper. Math. Z. 57, 86–93 (1952).

A subset $\mathfrak{o}$ of a field k is a half-order if it is closed under multiplication, contains 0 and ± 1, and there exists an element $z \in \mathfrak{o}$ such that $z(\mathfrak{o}+\mathfrak{o})\subset\mathfrak{o}$. If φ is an ordinary valuation of k, archimedian or not, then the set of "integral" elements $a \in k$ with $\varphi(a)\leqq 1$ is obviously a maximal half-order in k. The author's main result is the converse: every maximal half-order is the set of integral elements in some valuation. The corresponding result holds in skew fields for maximal half-orders which are invariant under inner automorphisms. *J. Tate* (Princeton, N. J.).

Jaeger, Arno. Gewöhnliche Differentialgleichungen in Körpern von Primzahlcharakteristik. Monatsh. Math. 56, 181–219 (1952).

Suppose that K is a field of characteristic $p\neq 0$, which admits a non-trivial differentiation D. Then there exist elements $z \in K$, called constants of order t, for which $D^n z=0$ for $0<n<p^t=m$, but $D^m z\neq 0$. In particular, there exists for each t an element $x=x_t \in K$ with $Dx_t=1$, $D^n x_t=0$, $1<n<p^t$, by means of which $D^n z$ can be identified with the nth (ordinary) derivative of z with respect to x_t. Then each $z \in K$ has, for given t, a unique representation $z=\sum_{\nu=0}^{m-1} z_\nu x_t^\nu$ where the z are constants of order at least equal to t; the coefficients z are given explicitly in terms of powers of x_t and the derivatives $D^\nu z$. After a discussion of the existence of a prolongation of D to inseparable extensions, the author formulates the problem of solving (by an element y in an extension K^* of K, which has a prolongation of D) an ordinary differential equation $f(y, Dy, \cdots, D^s y)=0$ by means of the aforementioned module-theoretic result as a problem of determining a subset M^* of the manifold of zeros which belongs to the ideal $M=(f_0(y_0, \cdots, y_s), \cdots, f_s(y_0, \cdots, y_s))$ in a polynomial ring of $s+1=p^t$ variables

$$\left(\text{where}\quad f(y, Dy, \cdots)=f\left(\sum_{\nu=0}^{s} y_\nu x^\nu, \cdots\right)=\sum_{\nu=0}^{s} f_\nu x^\nu\right)$$

over the subfield of K consisting all of elements which are of order at least equal to t (t is determined by the inequality $1<n<p^t$). In this manner the concepts of general zero, special zero, degree of transcendency of a zero are carried over to the theory of differential equations. Special attention is given to pure differential equations $D^n y=w \in K$, and a calculus of differential operators is developed for this purpose. A sample result is that the pure equation with $n=\sum_{\nu=0}^{r} n_\nu p^\nu$ (n_ν modulo p, $n_r\neq 0$) is solvable if and only if $D^k w=0$, $k=(p-n_\nu)p^\nu$ for all ν with $n_\nu\neq 0$. The solution y can then be determined explicitly and is found to depend on $a(n)=p^{r+1}-\prod_{\nu=0}^{r}(p-n_\nu)\geqq n$ parameters, which are constants of order at least equal to $r+1$. For the proof rather delicate considerations for the values of binomial coefficients modulo p (with an interesting generalization of Pascal's triangle) are needed; for example, the explicit determination of the degree of transcendency $d(\leqq a(n))$ of the manifold M of solutions belonging to a linear differential equation of formal order n depends on such evaluations. Apart from the examples of the hypergeometric and Bessel differential equations (the latter is not solvable for any $p\neq 2$) linear differential equations with constant coefficients are solved by operator methods. *O. F. G. Schilling* (Chicago, Ill.).

Jaeger, Arno. Partielle Differentialgleichungen in Körpern von Primzahlcharakteristik. Monatsh. Math. 56, 265–287 (1952).

Suppose that $\mathfrak{D}=(D_1, \cdots, D_\delta)$ is a regular multi-differentiation of the field K of prime characteristic, that is, there exists a set of elements x_μ such that $D_\nu x_\mu=\delta_{\nu\mu}$ and $D_\nu^\lambda x=0$ if $1<\lambda<p^t$, $\nu, \mu=1, \cdots, \delta$, for the component operators. [For the general theory of such multi-differentiations see Jaeger, J. Reine Angew. Math. **190**, 1–21 (1952); these Rev. **14**, 130.] In this paper the author generalizes the results of the preceding paper to partial differential equations $f(\mathfrak{D}^{t_1}y, \cdots, \mathfrak{D}^{t_n}y)=0$. It is shown how the results and details of the corresponding proofs carry over with some easy modifications which are of formal nature.
O. F. G. Schilling (Chicago, Ill.).

Osima, Masaru. On the Cartan invariants of algebras. Math. J. Okayama Univ. 2, 9–12 (1952).

Let A be an algebra with a unity over an algebraic number field K. Let J be an integral domain (Ordnung) in A, let $\mathfrak{p}$ be a prime ideal of the ring $\mathfrak{o}$ of algebraic integers in K, and set $\bar{A}=J/\mathfrak{p}J$. It is assumed that the irreducible representations of A in K are absolutely irreducible and a corresponding assumption is made concerning the irreducible representations of $\bar{A}$ in the field $\mathfrak{o}/\mathfrak{p}$. A formula is give which connects the Cartan invariants of $\bar{A}$, the Cartan invariants of A, and the decomposition numbers. This is a generalization of a result of the reviewer [Proc. Nat. Acad. Sci. U. S. A. **25**, 252–258 (1939)]. This generalization can also be obtained by the original method. *R. Brauer.*

Masuda, Katsuhiko. Direct decompositions of Galois algebras. Tôhoku Math. J. (2) 4, 122–130 (1952).

Suppose that G is a finite group and Ω a field whose characteristic is relatively prime to the characteristic of G and which contains the r irreducible characters $\{\chi\}=X$ of degrees f_χ. The author considers general (not necessarily associative) Galois algebras K/Ω, that is, algebras which admit the elements of G for right operators and which are isomorphic to the group ring $G(\Omega)$ as right $G(\Omega)$-modules [see Hasse, J. Reine Angew. Math. **187**, 14–43 (1949); these Rev. **11**, 576]. The aim is to obtain a criterion for K/Ω to

be a field, i.e., a criterion for the simplicity of a commutative associative Galois algebra. The author transforms this problem into a problem for so-called Galois structures, that is, double systems of r^2 matrices in Ω of degrees $f_\chi f_\psi = s$, $\{C_{\chi,\psi}; \chi, \psi \in X\}$, and of r irreducible representations $\{A_\psi; \psi \in X\}$. These systems which were also used by Hasse [loc. cit.] and which are given without ring operations, are called associated to a Galois algebra K/Ω with the factor basis W_ψ [see Hasse, loc. cit.] and transforming matrices $P_{\chi,\psi}$ if and only if $W_\psi{}^S = A_\psi(S)W_\psi$, $S \in G$, $\psi \in X$, and $W_\chi \times W_\varphi = P_{\chi,\varphi}^{-1} D P_{\chi,\varphi} C_{\chi,\varphi}$ where D is a diagonal matrix of s irreducible representations which are uniquely determined by χ and φ. The property of a Galois algebra K/Ω to be a direct sum of isomorphic algebras (transforms of each other by elements of G), which are Galois algebras of a subgroup $H \subset G$, is translated into a suitable (quite involved) definition of a Galois structure to be decomposed with respect to H. *O. F. G. Schilling* (Chicago, Ill.).

Wolf, Paul. Zur invarianten Kennzeichnung galoisscher Algebren mit vorgegebener Galoisgruppe. Abh. Math. Sem. Univ. Hamburg **18**, 179–195 (1952).

H. Hasse has shown [reference cited in the preceding review] that each Galois algebra K/Ω determines uniquely a class of factor sets $\{C_{\chi,\psi}\}$; moreover, he stated explicit conditions for these factor sets to belong to commutative, associative, semi-simple algebras K/Ω. The derivation of these conditions was based upon the explicit matricial representations Γ of the group G, i.e., reference to a fixed basis of the corresponding representation module was made. As in the more recent presentations of the theory of crossed products the author carries out the program to give a conceptual "invariant" derivation of the above mentioned conditions by replacing the representing matrices by the homomorphisms of the associated representation modules, from which they arise. If there corresponds to $1 \in G$, under a fixed isomorphism, the element $\Theta \in K$, then there belongs to each representation Γ with $S \rightarrow A_\Gamma(S)$ in the homomorphism ring H_Γ belonging to the representation module of Γ, a resolvent $w_\Gamma = \sum_{T \in G} A_\Gamma(T^{-1})\Theta^T$ with $w_\Gamma{}^S = A_\Gamma(S)w_\Gamma$, $S \in G$, which is uniquely determined to within a right (regular) factor in H_Γ. Using this fact it follows that the factor sets of Hasse can be introduced as the representations of the factor sets $c_{\Gamma,M} \in H_\Gamma \times H_M$ which appear as the right factors in $w_\Gamma \times w_M = w_{\Gamma \times M} \cdot c_{\Gamma,M}$ for irreducible Γ and M. Hasse's condition for commutativity, for example, assumes the simple form $c_{M,\Gamma} = c_{\Gamma,M}$. Finally the author also reformulates the condition that a system $c_{\Gamma,M}$ lead to a semi-simple algebra. Naturally the existence of a Galois algebra K for a given system depends on an explicit construction; for this purpose the author shows in detail how the passage leading back to Hasse's explicit formulas can be made.
O. F. G. Schilling (Chicago, Ill.).

Jacobson, N. Operator commutativity in Jordan algebras. Proc. Amer. Math. Soc. **3**, 973–976 (1952).

If B is a subset of a Jordan algebra A, we define $C_A(B)$ to be the set of all X in A such that $R_x R_b = R_b R_x$ for every b of B. Then $C_A(B)$ need not be a subalgebra of A. When A is special, semisimple, and of characteristic zero, $C_A(B)$ is shown to be a subalgebra of A. It is also a subalgebra if A has characteristic zero and B is a finite-dimensional semisimple subalgebra. The author also shows that if A is finite-dimensional of characteristic zero and A and B are both semisimple, then $C_A(B)$ is semisimple.
A. A. Albert (Chicago, Ill.).

Murakami, Shingo. On the automorphisms of a real semisimple Lie algebra. J. Math. Soc. Japan **4**, 103–133 (1952).

The author gives a complete determination of the group of outer automorphisms of a semi-simple real Lie algebra $\mathfrak{g}$ (i.e., the factor group of the group of all automorphisms of $\mathfrak{g}$ by its connected component of the identity). Let $\bar{\mathfrak{g}}$ be the complexification of $\mathfrak{g}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, and $\bar{\mathfrak{h}}$ its complexification, which is a Cartan subalgebra of $\bar{\mathfrak{g}}$. There corresponds to the real form $\mathfrak{g}$ of $\bar{\mathfrak{g}}$ an automorphism S of order 2 of $\bar{\mathfrak{g}}$, which may be taken to transform $\bar{\mathfrak{h}}$ into itself and which therefore permutes among themselves the roots of $\bar{\mathfrak{g}}$ with respect to $\bar{\mathfrak{h}}$. The group $\mathfrak{h}_R$ of elements of $\bar{\mathfrak{h}}$ at which all roots are real has a Euclidean metric; the author calls rotations the isometric transformations of $\mathfrak{h}_R$ which permute the roots among themselves, and inner rotations those which belong to the group generated by the reflections in the hyperplanes $\alpha = 0$ (α any root $\neq 0$) (i.e., the operations of what is often called the Weyl group). The group of outer automorphisms of $\mathfrak{g}$ is then well known to be the factor group of the group of all rotations by the group of inner rotations. Let $\alpha \rightarrow \alpha^*$ be the permutation of the roots defined by the rotation ρ produced by S. For any root $\alpha \neq 0$, let e_α be an element of $\mathfrak{g}$ belonging to α; then $S \cdot e_\alpha = \nu_\alpha e_{\alpha^*}$. The non-zero roots fall into three disjoint sets: Σ_1, the set of α's for which $\alpha^* = \alpha$, $\nu_\alpha = 1$; Σ_2, the set of β's for which $\beta^* = \beta$, $\nu_\beta = -1$; Σ_3, the set of ξ's for which $\xi^* \neq \xi$. The author shows that the group of outer automorphisms of $\mathfrak{g}$ is isomorphic to $\mathfrak{T}/\mathfrak{S}$, where $\mathfrak{T}$ and $\mathfrak{S}$ are the following groups of rotations. A rotation belongs to $\mathfrak{T}$ if and only if it commutes with ρ and maps each one of the sets Σ_i ($i = 1, 2, 3$) (or, which amounts to the same, Σ_1) into itself. For any root $\gamma \neq 0$, let $\sigma(\gamma)$ be the reflection in the hyperplane $\gamma = 0$. Then $\mathfrak{S}$ is generated by the operations $\sigma(\alpha)$, $\alpha \in \Sigma_1$, $\sigma(\xi + \xi^*)$, $\xi \in \Sigma_3$, ξ not orthogonal to ξ^* and $\sigma(\xi^*)\sigma(\xi)$, $\xi \in \Sigma_3$, ξ orthogonal to ξ^*. As an illustration of his method, the author determines anew all outer automorphisms of the real forms of the Lie algebras of type A_1. *C. Chevalley.*

Sul'din, A. V. On linear representations of Lie algebras over a field of characteristic $p > 0$. Doklady Akad. Nauk SSSR (N.S.) **82**, 529–531 (1952). (Russian)

The author states, but gives an incorrect proof of, the theorem: Let L be a Lie algebra over a field K of characteristic $p \neq 0$; then L admits a faithful completely reducible finite-dimensional representation—a statement which is not true for $p = 0$ or for algebraic Lie groups even when $p \neq 0$. The author's error consists in using the proposition that the regular representation of a semi-simple Lie algebra is completely reducible—which is false if $p \neq 0$. Nevertheless, the author's basic idea can be made into a correct proof which is very simple. The remarkable fact about the theorem in question is its simplicity compared to the situation in case $p = 0$. *G. D. Mostow* (Baltimore, Md.).

Zassenhaus, Hans. Über die Darstellungen der Lie-Algebren bei Charakteristik 0. Comment. Math. Helv. **26**, 252–274 (1952).

The general problem studied in this paper is that of the relation between the representations of a finite-dimensional Lie algebra L of characteristic 0 and those of an ideal (or subinvariant subalgebra) T of L. More specifically, the author determines those representations of T which are induced by representations of L. The main result is that a representation Γ of T is induced by one of L if and only if

$R(T)\cap[L, L]$ is represented by nilpotent transformations, where $R(T)$ is the radical of T. The necessity of this condition has been given elsewhere [N. Jacobson, Proc. Amer. Math. Soc. **2**, 105–113 (1951); these Rev. **14**, 241] and the sufficiency is of primary interest. Ado's theorem on the existence of faithful representations follows from this part of the result if T is taken as the center of L. The author concludes with an outline of a procedure to determine all the representations of L. *W. G. Lister.*

Zelinsky, Daniel. Linearly compact modules and rings. Amer. J. Math. **75**, 79–90 (1953).

L'auteur montre comment la notion de compacité linéaire permet d'éliminer les restrictions de dénombrabilité de sa récente étude des anneaux topologiques dont un système fondamental de voisinages de 0 est formé d'idéaux [Duke Math. J. **18**, 431–442 (1951); ces Rev. **12**, 795]. Il commence par généraliser la théorie des espaces vectoriels linéairement compacts aux modules sur un anneau topologique; la plupart des résultats s'étendent sans difficulté à ce cas plus général. Il applique ensuite ces propriétés, combinées aux techniques de son précédent article, pour montrer qu'un anneau semi-simple (au sens de Jacobson), ayant un système fondamental d'idéaux pour voisinages de 0, et linéairement compact, est produit d'anneaux simples discrets satisfaisant à la condition minimale. La même méthode lui donne aussi une généralisation de son résultat relatif aux anneaux commutatifs; un anneau commutatif, ayant un système fondamental de voisinages de 0 formé d'idéaux et linéairement compact est décomposable de façon unique en somme directe d'un anneau réduit à son radical et d'un nombre fini d'anneaux primaires. Il montre aussi qu'un anneau local complet est linéairement compact pour la topologie discrète, et qu'un anneau de valuation est linéairement compact pour la topologie discrète si et seulement s'il est maximal.

J. Dieudonné (Ann Arbor, Mich.).

Theory of Groups

***Tamari, Dov. Monoïdes préordonnés et chaînes de Malcev.** Thèse, Université de Paris, 1951. iv+81 pp. (mimeographed).

Malcev [Mat. Sbornik N.S. **6**(48), 331–336 (1939); **8**(50), 251–264 (1940); these Rev. **2**, 7, 128] has given necessary and sufficient conditions that a semigroup can be embedded in a group. In the present work the results of Malcev are extended to more general algebraic systems. The main result is the formulation of necessary and sufficient conditions that a given quasi-ordered semigroupoid S can be embedded in a quasi-ordered groupoid G. A quasi-ordering (préordre) is a reflexive, transitive relation, and it is understood that the quasi-ordering of G is to be an extension of that of S. The results of Malcev are therefore generalized in two ways; first by considering semigroupoids rather than semigroups, that is, systems in which the product of two elements is not always defined, and second, by imposing a quasi-ordering on S which, as well as the composition function, must be extended to G. The necessary and sufficient conditions obtained are too complex to state in a brief review. The results of Malcev, Ore [Ann. of Math. (2) **32**, 463–477 (1931)] and Doss [Bull. Sci. Math. (2) **72**, 139–150 (1948); these Rev. **10**, 591] are obtained as special cases.

D. C. Murdoch (Vancouver, B. C.).

Cohn, P. M. Generalization of a theorem of Magnus. Proc. London Math. Soc. (3) **2**, 297–310 (1952).

If G is a group, elements $\sum a_s s$, $s \in G$, a_s an integer, form the group ring R whose additive group is free abelian. Group rings whose additive group is the direct product of cyclic groups are also considered here. Elements in R with $\sum a_s=0$ form the "Magnus ring" A, which is a two-sided ideal in R. The Magnus ring A is characterized essentially by $R/A=J$, the additive group of integers. With the bracket operation $[x, y]=xy-yx$ we may regard R also as a Lie ring. Let A^n be the nth associative power of A and $A^{[n]}$ its nth Lie power. Then if $G=G_1, G_2, \cdots$ is the lower central series of G it is shown that $G_n/G_{n+1}=A^{[n]}/A^{n+1}$, the quotient on the right being that of the additive groups of the corresponding rings. This is the generalization of the theorem proved by Magnus for free groups [J. Reine Angew. Math. **177**, 105–115 (1937)]. The proof depends on extending the Birkhoff-Witt process for embedding a Lie ring in an associative ring to the case in which the additive group is the direct sum of cyclic groups. *Marshall Hall.*

Szele, T., and Szendrei, J. On abelian groups with commutative endomorphism ring. Acta Math. Acad. Sci. Hungar. **2**, 309–324 (1951). (Russian summary)

If an Abelian torsion-group has a commutative ring of endomorphisms, then every endomorphic image of it (and hence every subgroup of it) is fully invariant. The group must therefore be the (restricted) direct sum of cyclic p-groups or groups of type (p^∞), at most one for each prime number p. Another description of such a group is that it is a subgroup of the group of all rotations of finite order of the circle, or of the additive group of rationals (mod 1). In the case of mixed Abelian groups, a characterisation of those with commutative ring of endomorphisms is not quite so simple. Let G be a mixed group with commutative ring of endomorphisms, and T its torsion-group. Then T is again locally cyclic but contains no subgroups of type (p^∞). Moreover for each prime number p which occurs as actual order in T the factor-group G/T is closed for p: $pG/T=G/T$. If in addition G does not possess any elements of infinite height with respect to these same prime numbers p, then G lies between the restricted and the unrestricted direct sum extended over the p-primary components of T. These conditions (together with the condition that every endomorphic image of G is fully invariant) are sufficient as well as necessary for G to have a commutative ring of endomorphisms.

The authors conjecture that the single condition that every endomorphic image be fully invariant is necessary and sufficient for G to have a commutative ring of endomorphisms; and that such a group G has at most the power of the continuum. If these conjectures turn out to be true, then every group with a commutative ring of endomorphisms is a subgroup of the additive group of the reals (mod 1). *K. A. Hirsch* (London).

Azleckiĭ, S. P. On the Sylow rank and length of the principal and composition series of a finite group. Mat. Sbornik N.S. **31**(73), 359–366 (1952). (Russian)

This paper continues the study of Sylow rank initiated by the author [Mat. Sbornik N.S. **28**(70), 461–466 (1951); these Rev. **12**, 799]. The Sylow rank of a group cannot exceed the length of a principal series. If the Sylow rank is equal to the length of a composition series, the group is the direct product of simple groups. If a solvable group has Sylow rank equal to the length of a principal series, the group is a special group. If a solvable group has Sylow rank

equal to the length of a composition series, the group is abelian of square-free order. In the second section of the paper, a description is given of groups with composition series of length 2. Seven types are described, five of these completely. *J. L. Brenner* (Pullman, Wash.).

Pavlov, P. P. Sylow p-subgroups of the full linear group over a simple field of characteristic p. Izvestiya Akad. Nauk SSSR. Ser. Mat. 16, 437–458 (1952). (Russian)

The author considers the set of all matrices of order n with zeros below the diagonal and unity across the diagonal, over the prime field F of characteristic p. These matrices, whose number is $p^{n(n-1)/2}$, constitute a Sylow subgroup G of the group of all regular matrices over F. Denote by e_{ik}, $i, k=1, \cdots, n$, the matric units and consider the regular matrices $\epsilon_{ik}=1+e_{ik}$. It is shown that the matrices $\epsilon_{i,i+1}$ $(i=1, \cdots, n-1)$ constitute an irreducible system of generators for G. The author determines a fundamental set of relations among the ϵ's in terms of certain commutators, from which all relations of the form $\epsilon_{i_1 i_1+1}^{m_{i_1}} \cdots \epsilon_{i_s i_s+1}^{m_{i_s}}=1$ may be derived. The author then proceeds with a detailed study of the classes of conjugate elements in G. One of his results is: If $\alpha=1+\sum a_{ik}e_{ik}$, $\alpha \in G$, $a_{ik}\neq 0$, then the number of elements in the class of α is at least $p^{n+i-k-1}$. These results are applied to the study of the structure of the automorphism group A of G. Consider the group V, generated by the so-called "extremal" automorphisms σ_1 and σ_2 that are defined by setting $\epsilon_{12}^{\sigma_1}=\epsilon_{12}\epsilon_{2n}$, $\epsilon_{n-1,n}^{\sigma_2}=\epsilon_{n-1,n}\epsilon_{1,n-1}$ (the remaining $\epsilon_{i,i+1}$ are unchanged) and the group Z of the "central" automorphisms τ_α that are defined by setting $\epsilon_{\alpha,\alpha+1}^{\tau_\alpha}=\epsilon_{\alpha,\alpha+1}\epsilon_{1n}$ (the remaining $\epsilon_{i,i+1}$ are unchanged). The product $P=V\cdot Z$ is direct and constitutes an abelian subgroup of A of order p^{n-1}. If I is the group of inner automorphisms of G, then the product $I\cdot P$ is "semi-direct" (i.e., I is transformed into itself by the elements of P) and the order of this subgroup of A is $p^{(n^2+n-4)/2}$. Transforming G with a diagonal regular matrix one obtains a so-called diagonal automorphism of G. There are exactly $(p-1)^{n-1}$ such automorphisms, forming a group D, and the product $(I\cdot P)D$ is semi-direct with respect to $I\cdot P$. Finally, the transformation $\epsilon^{\tau}_{\alpha,\alpha+1}=\epsilon_{n-\alpha,n-\alpha+1}$ $(\alpha=1, \cdots, n-1)$, called a reflection, is an automorphism of order 2, and it is found that A is the semi-direct product $A=[(I\cdot P)D]\cdot\{\tau\}$, so that the order of A is

$$2(p-1)^{n-1}\cdot p^{(n^2+n-4)/2}.$$

The author states that his results may be generalized to the case where F is any field with a finite number of elements. Some of the subgroups of A that are considered in this paper were previously mentioned by Dubisch and Perlis [Amer. J. Math. 73, 439–452 (1951); these Rev. 12, 798] in the more general case where F is an arbitrary field.

J. Levitzki (Jerusalem).

Gaschütz, Wolfgang. Über den Fundamentalsatz von Maschke zur Darstellungstheorie der endlichen Gruppen. Math. Z. 56, 376–387 (1952).

Let $\mathfrak{g}$ be a finite group, Ω a set (or a ring), and $\mathfrak{a}$ a $\mathfrak{g}$-Ω-(double-)module. When every $\mathfrak{g}$-Ω-module $\mathfrak{b}$ directly decomposable into $\mathfrak{a}$ and an Ω-module can be directly decomposed into $\mathfrak{a}$ and a $\mathfrak{g}$-Ω-module, then we say that $\mathfrak{a}$ is an M_u-module. If every $\mathfrak{g}$-Ω-module $\mathfrak{b}$ containing a $\mathfrak{g}$-Ω-module $\bar{\mathfrak{a}}$ isomorphic to $\mathfrak{a}$, such that $\mathfrak{b}$ is directly decomposed into $\bar{\mathfrak{a}}$ and an Ω-module, may be directly decomposed into $\bar{\mathfrak{a}}$ and a $\mathfrak{g}$-Ω-module, then we call $\mathfrak{a}$ an M_o-module. It is proved that $\mathfrak{a}$ is an M_u-(or M_o-)module if and only if there exists an Ω-endomorphism α of $\mathfrak{a}$ with $\sum_{g\in\mathfrak{g}} g\alpha g^{-1}=1$. In the proof of the second half, use is made of the $\mathfrak{g}$-Ω-module of all mappings of $\mathfrak{g}$ into $\mathfrak{a}$. The Maschke-Schur theorem is an immediate consequence of the theorem. The theorem is applied to monomial representations of $\mathfrak{g}$ in a commutative ring, as well as to the universal splitting of factor sets (or cocycles in general) of $\mathfrak{g}$ in a $\mathfrak{g}$-module. *T. Nakayama.*

Murnaghan, F. D. On the decomposition of tensors by contraction. Proc. Nat. Acad. Sci. U. S. A. 38, 973–979 (1952).

This paper is an alternative to G. Racah's method [Rev. Modern Physics 21, 494–496 (1949); these Rev. 11, 542]. Racah has developed formulas which reduce the irreducible representations of the 3- and 4-dimensional linear groups over their rotation subgroups. His formulas determine moreover the number of times that any individual irreducible representation of the rotation groups appears in the reduction of the given irreducible representation of the containing linear group. The author shows in the present paper how to obtain the desired reduction, not only over the rotation subgroups but also over the orthogonal subgroups. The method may be applied not only to the 3- and 4-dimensional orthogonal groups but to orthogonal groups of any dimension. The desired reduction is obtained with very little labor by a skillful denotation of the typical irreducible representations.

M. Pinl (Dacca).

Itô, Seizô. Unitary representations of some linear groups. II. Nagoya Math. J. 5, 79–96 (1953).

Let G_n be the group of all euclidean motions in n-space E_n. Let V be the normal subgroup of all translations and let $K\cong SO(n)$ be the subgroup leaving a point of E_n fixed. Then every element of G_n is uniquely of the form vk where $v \in V$ and $k \in K$. It follows from a general theorem of the reviewer [Proc. Nat. Acad. Sci. U. S. A. 35, 537–545 (1949); Ann. of Math. (2) 55, 101–139 (1952); these Rev. 11, 158; 13, 434] that every irreducible unitary representation of G_n is either trivial on V and hence defined by an irreducible representation of K or else is defined uniquely by a positive real number r and an irreducible representation of $SO(n-1)$. Let w be any member of the character group of V which is at a distance r from 0 and let K_w be the subgroup of all $k \in K$ such that $w(k^{-1}vk)=w(v)$ for all $v \in V$. Then $K_w\cong SO(n-1)$ and if L is any irreducible representation of K_w, the mapping $vk\to w(v)L_k$ is a representation of the subgroup of all vk with $k \in K_w$. The representation of G_n induced by this representation of a subgroup is the irreducible representation associated with the number $r>0$ and the irreducible representation L of $K_w\cong SO(n-1)$.

In Theorem 1 of the paper under review the author gives an independent discussion of the irreducible representations of G_n which describes them in a manner somewhat more computational and less conceptual than that outlined above. In particular, no use is made of the notion of induced representation. In Theorem 2 he determines the most general cyclic representation of G_n. As a corollary he shows (Theorem 3) that every positive definite function is an integral of elementary positive definite functions. The arguments depend to some extent on those given in an earlier paper [Nagoya Math. J. 4, 1–13 (1952); these Rev. 13, 722] in which the case $n=2$ is treated. *G. W. Mackey.*

Matsushima, Yozô. Some remarks on the exceptional simple Lie group F. Nagoya Math. J. 4, 83–88 (1952).

It is known that a compact form of the simple exceptional Lie algebra F_4 can be identified as the algebra of derivations

of J, the exceptional simple Jordan algebra, which consists of all 3×3 Hermitian matrices with Cayley numbers as coefficients. The group F_4 can also be regarded as a group acting on the projective plane P over the Cayley numbers, the points of P being regarded as the irreducible idempotents of J. The author proves that certain subgroups of F_4 which are related to the representation of F_4 as a subgroup of the projective group on P are connected and simply connected. Let E_i denote the 3×3 matrix (a_{pq}) in J such that $a_{pq}=0$ if p or $q\neq i$, and $a_{ii}=1$. Let N, H_i denote the subgroups of F_4 (regarded as a group of automorphisms of J) which keep $\{E_1, E_2, E_3\}$ and E_i, respectively, fixed. The author proves that N and H_i are connected and isomorphic to the simply connected covering groups of $SO(8)$ and $SO(9)$ respectively. The result concerning H_i has been announced by A. Borel [C. R. Acad. Sci. Paris **230**, 1378–1380 (1950); these Rev. **11**, 640]. *G. D. Mostow* (Baltimore, Md.).

Lazard, Michel. Sur les groupes analytiques dans les modules filtrés. C. R. Acad. Sci. Paris **235**, 1465–1467 (1952).

Let L be a vector space over the field of rational numbers which is given a topological structure by means of a distance which satisfies a triangle inequality of the p-adic type; L is assumed to be complete. The author defines axiomatically a class of functions on L^m to L, called analytic functions; these functions may be represented as sums of series of homogeneous functions, where a homogeneous function of degree k with respect to one of its arguments x is a function such that $f(nx)=n^kf(x)$ for any integer n. Now, assume that L has a structure of group, the mapping $(x,y)\rightarrow xy^{-1}$ being analytic. Then the author shows that it is possible to introduce canonical coordinates and to establish the validity of the Campbell-Hausdorff formulas. These results generalize those of Dynkin [Uspehi Matem. Nauk (N.S.) **5**, no. 1(35), 135–186 (1950); these Rev. **11**, 712]. *C. Chevalley* (New York, N. Y.).

Rådström, Hans. Convexity and norm in topological groups. Ark. Mat. **2**, 99–137 (1952).

Let G be a topological group and let C be the totality of closed subsets of G. A Hausdorff topology is introduced into C by defining as open neighborhood of H_0, element of C, the totality of elements H of C such that $H_0\subset NH$, $H\subset NH_0$ where N is any open neighborhood of the identity e. A one-parameter semigroup in G is defined as a mapping ϕ: $\delta\rightarrow A_\delta$ of the non-negative reals into the set of subsets of G such that (1) $A_{\delta_1}A_{\delta_2}=A_{\delta_1+\delta_2}$ and (2) the restriction of ϕ to some closed interval $0\leqq\delta\leqq\alpha$ $(\alpha>0)$ is a continuous mapping into C. This restriction is shown to be a homeomorphism if α is sufficiently small provided that ϕ is non-constant. Moreover, for a number of purposes it can be assumed that $\phi(0)=\{e\}$. The main result is that in a Lie group G, every one-parameter semigroup ϕ such that $\phi(0)=\{e\}$ has a unique infinitesimal generator K which is a compact convex set in the Lie algebra g of G. To say that K generates ϕ means that $\phi(\gamma)=\lim_{\delta\rightarrow 0}(f(\delta K))^{\gamma/\delta}$ where f is the exponential mapping and δK consists of the elements of K multiplied by the scalar δ. Every compact set (not necessarily convex) in g generates a ϕ and in fact generates the same ϕ as its convex hull. The compact convex set which generates a given ϕ is $K=\lim_{\delta\rightarrow 0}(1/\delta)f^{-1}\phi(\delta)$. When G is abelian, $\phi(\delta)=f(\delta K)$. A topological group is called normed if its topology is defined by a left-invariant metric such that the closed spheres S_δ of radius δ about the identity e determine a one-parameter semigroup ϕ: $\delta\rightarrow S_\delta$. In order that a left-invariant metric define a normed group it is necessary and sufficient that it be convex [in a sense slightly different from that of Menger, Math. Ann. **100**, 75–163 (1928)]. Every locally compact normed group is separable, metric, connected and locally connected. Various classes of topological groups including connected Lie groups, are shown to be normable. *P. A. Smith* (New York, N. Y.).

NUMBER THEORY

★**Hasse, Helmut. Vorlesungen über Zahlentheorie.** Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete. Band LIX. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1950. xii+474 pp. 42.00 DM; bound, 45.00 DM.

This book is a welcome addition to this famous series and seems destined to become one of its most significant volumes. It is more elementary and along more classical lines than the author's Zahlentheorie [Akademie-Verlag, Berlin, 1949; these Rev. **11**, 580]; in particular, it contains no valuation theory. However, it is a book for serious students of mathematics and not for those interested in numerical curiosities. Although the material discussed is classical, it is approached from a present-day point of view. Thus the author does not hesitate to use the language of modern algebra when this is the natural thing to do; this is a refreshing contrast to many books on number theory. Actually not much specific knowledge is required of the reader, either in algebra or analysis, but a considerable amount of mathematical maturity is called for.

The range of subject matter covered by this work is comparatively narrow, but those topics which are discussed are treated very thoroughly. Much space is devoted to explaining the background, motivation, and significance of the theorems proved, and many theorems are proved in more than one way. In fact, one often has the feeling that the author just cannot bear to hide the least bit of pertinent information from the reader. This thoroughness is very instructive, but in many parts of the book progress is very slow as a result. Some idea of this extreme thoroughness can be obtained by comparing the part of this book devoted to Dirichlet's theorem on prime numbers with the corresponding part of Landau's work of the same title. Landau disposes of Dirichlet's theorem in 18 pages, while Hasse discourses on the subject for 102 pages.

The first part of the book (79 pages) is entitled "Foundations" and deals with the elementary arithmetic properties of the rational integers. The unique factorization theorem for the rational integers is proved by Zermelo's inductive argument and then divisibility, greatest common divisor, and least common multiple are treated with the aid of factorization into primes. The Euclidean algorithm is introduced to show how the greatest common divisor of two integers may be computed and also to provide another proof (the usual one) of the unique factorization theorem. Then follows a discussion of congruence, the residue class ring, and the prime residue class group modulo a positive integer, including a proof of Fermat's theorem and a treatment of

Euler's function. This part concludes with the determination of the structure of the prime residue class groups.

The second part (87 pages) is entitled "Quadratic residues". After reducing the general discussion of quadratic residues to the case where the modulus is an odd prime number, the author develops the criteria of Legendre, Euler, and Gauss for quadratic residuacity modulo an odd prime. Two proofs are given for the quadratic reciprocity law, namely, Frobenius' modification of Gauss's third proof and a proof using congruences in cyclotomic fields and Gaussian sums. The Jacobi and Kronecker extensions of the Legendre symbol are introduced and discussed thoroughly from the standpoint of the theory of residue characters. The second part concludes with a long section on the distribution of quadratic residues modulo a prime number p; this includes Jacobstahl's work on the number of integers r between 1 and $p-3$ for which $(r|p)$, $(r+1|p)$, and $(r+2|p)$ have prescribed values.

The third part (102 pages) is entitled "Dirichlet's prime number theorem" and, as mentioned earlier, is devoted to a thorough analysis of the classical proof of Dirichlet's theorem that, if r and m are coprime positive integers, there are infinitely many prime numbers congruent to r modulo m. After presenting the classical elementary proofs for the cases $r=1$ and $r=m-1$, the author gives a rough sketch of the proof to be used in the general case and discusses its relation to Euler's proof that there are infinitely many primes. This is followed by a detailed exposition of the theory of characters of finite abelian groups and of residue characters in particular. Assuming the non-vanishing of $L(1, \chi)$ for the non-principal residue characters, the author then gives the usual proof of Dirichlet's theorem. Finally there is a long but very illuminating discussion of the various methods of proving that $L(1, \chi)\neq 0$. It is unfortunate that this work was written before the appearance of the various elementary proofs of Dirichlet's theorem, for this third part would have been much more valuable if these proofs were included in the discussion.

The fourth and final part of the book (198 pages) is entitled "Quadratische Zahlkörper". This begins with the usual material on integers, divisibility, discriminant, integral bases, and units. The calculation of the fundamental unit of a real quadratic field by means of continued fractions is treated in detail. Quadratic fields with unique factorization into prime numbers are discussed as a means of foreshadowing the more general theory. The arithmetic of general quadratic fields is then discussed from the standpoint of the theory of divisors, ideals being introduced only after the general unique factorization theorem is given in terms of divisors. After proving the finiteness of the class-number, the author gives a detailed discussion of its calculation. He explains how the splitting law for quadratic fields and the quadratic reciprocity law are special cases of more general theorems in class-field theory. Finally the work closes with a systematic treatment of Gaussian sums.

The book is almost completely free of misprints, but the reviewer noticed one mathematical error. On page 225 the author asserts that a set M of prime numbers has Dirichlet density δ, that is,

$$(*) \qquad \lim_{s\to 1+0}\left(\log\frac{1}{s-1}\right)^{-1}\sum_{p\varepsilon M} p^{-s}=\delta,$$

if and only if it has natural density δ, that is,

$$(**) \qquad \lim_{x\to\infty} x^{-1}\log x \sum_{p\varepsilon M,\, p\leq x} 1=\delta.$$

Actually (**) is a much stronger property than (*), that is, (**) implies (*), but (*) does not imply (**). For example, the set of prime numbers p such that $[\log p]$ is an even integer has Dirichlet density $\frac{1}{2}$, but does not have a natural density. It can be shown that (*) is equivalent to

$$\lim_{x\to\infty}(\log\log x)^{-1}\sum_{p\varepsilon M,\, p\leq x} p^{-1}=\delta.$$

[Cf. Wintner, The theory of measure in arithmetical semigroups, Baltimore, 1944, §31; these Rev. 7, 367; Hardy, Divergent series, Oxford, 1949, Theorem 108; these Rev. 11, 25.]

Although the heaviness of style will undoubtedly discourage some readers, this book must be rated as a masterpiece. *P. T. Bateman* (Urbana, Ill.).

***Kraitchik, M. Introduction à la théorie des nombres.** Gauthier-Villars, Paris, 1952. vii+202 pp. 2400 francs.

This is a comprehensive exposition of several elementary topics in the theory of numbers, intended partly for amateur mathematicians interested in this field. Thus there is much more attention to numerical cases than is customary, and this is hinted in the dedication: "Ce livre est dédié à tous les amis des nombres, à tous ceux, pour qui tout entier positif est un ami personnel". The theory of numbers is treated as a subject apart and self-contained; connections with other parts of mathematics, such as that the simple Fermat and Euler theorems are special cases of theorems on finite groups, are not treated.

The chapter headings are as follows: Introduction; The indicator (Euler's totient or ϕ function); Congruences; General properties of congruences; Binomial and exponential congruences; Applications to binomial equations; Linear congruences; The reciprocity law; Certain congruences dependent on the reciprocity law; Residues of large numbers; Cycles; On the equation $x^2-y^2=N$; Algebraic factorization; Direct proofs of primality.

There are many numerical tables: primes; factors of $2^n\pm1$ and $10^n\pm1$; n-ic residues of small primes with $n=2, 3, 4, 5, 6, 7, 11$; congruence properties of divisors of $x^2\pm Dy^2$ for various values of D; solutions x for $x^2-y^2\equiv N \pmod p$ for various $p<100$.

The style will probably be attractive to many beginners, but they may be sorely tried at one or two points, as, for example, on p. 15 where the following two problems occur in the same set with no indication that the first is a celebrated unsolved problem, contrasting sharply with the second simpler one. 1) There exist infinitely many pairs of primes with difference 2. 2) The product of two numbers equals the product of their g.c.d. and their l.c.m.

I. Niven (Eugene, Ore.).

Waterson, A. On the sum of the r-th powers of the first n integers. Edinburgh Math. Notes 38, 9–13 (1952).

Set $S_n^r=1^r+2^r+\cdots+n^r$. The easily proved identity $(n+1)^{m+1}-1=S_n^0+\binom{m+1}{1}S_n^1+\cdots+\binom{m+1}{m}S_n^m$ is written down for $m=1, 2, \cdots, r$ and the resulting equations are solved for S_n^r in terms of determinants. This gives an explicit expression for S_n^r which can be transformed in various ways and in particular may be written in the form of a polynomial in n. The final result can be identified with the usual formula involving Bernoulli numbers.

H. W. Brinkmann (Swarthmore, Pa.).

Tallqvist, Hj. Die Potenzsummen der ganzen Zahlen. Soc. Sci. Fenn. Comment. Phys.-Math. 15, no. 1, 7 pp. (1951).

Tallqvist, Hj. Produktsummen der ganzen Zahlen. Soc. Sci. Fenn. Comment. Phys.-Math. 16, no. 5, 5 pp. (1952).

In the first of these notes explicit formulas are given for $\sum_{k=1}^{n} k^r$ and $\sum_{k=0}^{n}(2k+1)^r$, $r=1, 2, \cdots, 12$. Numerical values of these sums are tabulated for $n=1, 2, \cdots, 12$. In the second note the author derives formulas for sums of the form $\sum k_1k_2\cdots k_r$ $(r=2, 3, 4)$ where the k_i run through the integers from 1 to n; the various possibilities of equalities among the k_i are considered. Numerical values are again tabulated for $n=1, 2, \cdots, 12$; a spot check revealed a few errors in these calculations. *H. W. Brinkmann.*

Miller, J. C. P. The sum of the integral parts in an arithmetical progression. Math. Gaz. 36, 234–243 (1952).

Let $S(N; c, a)=\sum_{r=0}^{N-1}[rc+a]$ where $[x]$ denotes the integral part of x as usual; the paper deals with a recursive procedure for evaluating this sum. Thus for $a=0$ we set $N_1=[Nc]$, $1/c=d_1-c'$ $(0<c'<1)$ and obtain $S(N; c, 0)=N_1(N+1-\frac{1}{2}d_1(N_1+1))+S(N_1; c', 0)$. The evaluation of $S(N; c, 0)$ then depends upon a certain continued fraction expansion of c. Certain variations of this formula are also given and these lead to the use of different continued fraction expansions of c. The case $a\neq 0$ is treated in a similar manner. *H. W. Brinkmann* (Swarthmore, Pa.).

Obláth, Richard. Untere Schranken für Lösungen der Fermatschen Gleichung. Portugaliae Math. 11, 129–132 (1952).

Lower bounds are given for x, y, and z in the equation $x^n+y^n=z^n$ where n is an odd prime. In the so-called first case (n does not divide xyz) the author gives

$$x>4n^2+1, \qquad y>2(4n^2+1), \qquad z\geqq 2(10n^2+1).$$

Here the author could have made use of a theorem of Peter Barlow and obtained practically the nth powers of the above lower bounds. In fact, Inkeri [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 33 (1946); these Rev. 9, 411] has shown that $x>n(30n^2+1)^n$. In the second case (n divides x) the author uses a theorem of Vandiver, which states that n^3 divides x, to show that $z>n^{3n-1}$. Since $n>619$ in the second case and $n>253747889$ in the first case, x, y, and z are extremely large in either case.

The author calls attention to a statement of Dickson to the effect that R. Sauer proved that neither x, y, nor z can be "a power of a prime." Apparently the theorem has not been really proved. *D. H. Lehmer* (Los Angeles, Calif.).

Cassels, J. W. S. On the equation $a^x-b^y=1$. Amer. J. Math. 75, 159–162 (1953).

It was shown by the reviewer [same J. 74, 325–331 (1952); these Rev. 13, 822] that the equation of the title has at most one solution if $(a, b)\neq(3, 2)$, and this solution was effectively specified. The present author gives a simpler proof of the following slightly stronger theorem: Suppose that $x, y, a>1$, $b>1$ are positive integers, and the equation is not $3^2-2^3=1$. Suppose also that ξ, η are the least positive solutions of $a^\xi\equiv 1 \pmod{B}$, $b^\eta\equiv -1 \pmod{A}$, where A and B are the products of the odd prime divisors of a, b respectively. Then $x=\xi$, $y=\eta$, except that $x=2$, $y=1$ may occur if $\xi=\eta=1$ and $a+1$ is a power of 2. In the course of the proof it is also shown that any prime divisor of y which is less than x divides a, and vice versa. *W. J. LeVeque.*

Mordell, L. J. The congruence $ax^3+by^3+c\equiv 0 \pmod{xy}$, and integer solutions of cubic equations in three variables. Acta Math. 88, 77–83 (1952).

It is proved that the congruence of the title has an infinity of solutions for which $(cx, y)=1$; the solutions obtained are polynomials in a, b, c. The result also holds if ax^3 and by^3 are replaced by any polynomials $f(x)$ and $g(y)$ respectively, with integral coefficients and $f(0)=g(0)=0$. With this it is proved that $z^2-(27abc)^2=ab^2x^3+y^3$ has an infinity of integral solutions, a, b, c being integers; a similar result was obtained earlier by the author [J. London Math. Soc. 17, 199–203 (1942); these Rev. 4, 265; 6, 334]. These results support a conjecture of the author that if $F(x, y, z)=0$ is a cubic with integral coefficients, not representing a cone in three-space, and if $F-a$ is irreducible for all integers a, then one integral solution of $F=0$ implies an infinity of such solutions. However, the author added in proof two counterexamples to this conjecture; for example, $x^3+y^3+z^3+4xyz=1$ is said to have only solutions typified by $y=z=0$. Remark by reviewer: the author seems to assume in the proofs that certain coefficients are not zero, without formal hypotheses in the statements of the theorems. *I. Niven.*

Roth, K. F. On certain sets of integers. J. London Math. Soc. 28, 104–109 (1953).

Denote by $A(x)$ the maximum number of integers not exceeding x, no three of which form an arithmetic progression. The author proves $A(x)<cx/\log\log x$. In a previous paper [C. R. Acad. Sci. Paris 234, 388–390 (1952); these Rev. 13, 724] the author proved that $A(x)=o(x)$; the method of the present paper is a sharpening of the previous one. *P. Erdös* (Los Angeles, Calif.).

Iseki, Kanesiroo. A divisor problem involving prime numbers. Jap. J. Math. 21 (1951), 67–92 (1952).

The paper deals with the function $D(x)=\sum_{p<x}d(x-p)$, where x denotes a positive integer, p a prime, and $d(n)$ the number of positive divisors of n. It is shown that

$$A_1\frac{\phi(x)}{\log\log x}\leqq D(x)\leqq A_2\phi(x) \qquad (x\geqq 3)$$

and, on the extended Riemann hypothesis,

$$D(x)=\phi(x)\prod_{p\nmid x}\left(1+\frac{1}{p(p-1)}\right)+O\left(\frac{\log\log x}{\log x}\phi(x)\right),$$

where A_1 and A_2 are positive constants, and $\phi(x)$ is Euler's function. A refinement of a theorem of Titchmarsh [Rend. Circ. Mat. Palermo 54, 414–429 (1930)] concerning the function $D_y(x)=\sum_{l<p\leqq x}d(p-l)$ is also obtained.

T. Estermann (London).

Ricci, Giovanni. Funzioni aritmetiche: proprietà asintotiche—aritmetica analitica. Archimede 4, 1–7, 98–104, 148–155 (1952).

Expository paper.

Williams, G. T. A new method of evaluating $\zeta(2n)$. Amer. Math. Monthly 60, 19–25 (1953).

The formula

$$\zeta(2)\zeta(2n-2)+\zeta(4)\zeta(2n-4)+\cdots+\zeta(2n-2)\zeta(2) = (n+\tfrac{1}{2})\zeta(2n)$$

is proved by an elementary method. It seems to have some relation to the more complicated formulae obtained by the reviewer [Proc. London Math. Soc. (2) 26, 1–11 (1926)].

Another result is

$$2\sum_{\nu=1}^{\infty}\frac{1}{\nu^n}\left(1+\tfrac{1}{2}+\cdots+\frac{1}{\nu-1}\right)=n\zeta(n+1)$$
$$-\{\zeta(2)\zeta(n-1)+\zeta(3)\zeta(n-2)+\cdots+\zeta(n-1)\zeta(2)\}.$$

E. C. Titchmarsh (Oxford).

Fawaz, A. Y. On an unsolved problem in the analytic theory of numbers. Quart. J. Math., Oxford Ser. (2) 3, 282–295 (1952).

The title refers to the two conflicting conjectures by Pólya and Ingham regarding the number-theoretic function $L(x)$ [Fawaz, Proc. London Math. Soc. (3) 1, 86–103 (1951); these Rev. 13, 327]. In the cited paper the author obtained an explicit formula for $L_0(x)$ ($L_0(x)$ differs from $L(x)$ only when x is an integer, the difference then being $\pm\frac{1}{2}$) in which appears the integral

$$I(x)=\frac{1}{2\pi i}\int_{(a)}\frac{\zeta(2s)}{s\zeta(s)}x^s ds\quad (0<a<\tfrac{1}{2}).$$

In the present paper he proves that

$$\liminf_{x\to\infty} x^{-\frac{1}{2}}L(x)=-\limsup_{x\to 0+} x^{-\frac{1}{2}}I(x),$$
$$\limsup_{x\to\infty} x^{-\frac{1}{2}}L(x)=-\liminf_{x\to 0+} x^{-\frac{1}{2}}I(x),$$

so that information about $I(x)$ can be applied to $L(x)$.

W. H. Simons (Vancouver, B. C.).

Venkataraman, C. S. Modular multiplicative functions. J. Madras Univ. Sect. B. 19, 69–78 (1950).

A function $F(M, N)$ (M, N positive integers) is called modular if it is multiplicative (that is,

$$F(M_1, N_1)F(M_2, N_2)=F(M_1M_2, N_1N_2)$$

whenever $(M_1N_1, M_2N_2)=1$), and if furthermore

$$F(M, N_1)=F(M, N_2)$$

whenever $N_1\equiv N_2 \pmod M$). An example is Ramanujan's sum $C_M(N)$. The author remarks that a modular function is: (i) multiplicative in M and N; (ii) multiplicative in M alone; (iii) it satisfies $F(M, N)=F(M, g)$, where $g=(M, N)$. He investigates to what extent these properties are characteristic for such functions, and finds: the combinations (i)+(iii) and (ii)+(iii) are both sufficient for modularity, but (i)+(ii) is not.

Furthermore the author proves that several products and convolutions of modular functions are again modular. For instance, if F and ψ are modular, then

$$\sum_{d|M}F(M/d, N)\psi(d, N)\quad\text{and}\quad\sum_{d|(M,N)}F(M/d, N/d)$$

are also modular. Finally, sums of the form $\sum F(M, Nh)$, where h runs through a complete set of residues mod M, are evaluated.

N. G. de Bruijn (Amsterdam).

Val'fiš, A. Z. Elementary solution of Pell's equation. Akad. Nauk Gruzin. SSR. Trudy Mat. Inst. Razmadze 18, 116–132 (1951). (Russian. Georgian summary)

The author (whose name is also spelled Walfisz) claims that all previous treatments of Pell's equation $x^2-ky^2=1$ depend tacitly on limiting processes as they either operate in the field of real numbers or use cyclotomy (Kreisteilung). Here he shows how known proofs can be modified to avoid this dependence. To prove the existence of solutions he first proves that for any positive integer m there is a rational r with $k<r^2<k+m^{-2}$ and then uses the Schubfachprinzip to show the existence of integers x, y with $|x-ry|<m^{-1}$, $0<y\leq m$. Then $|x^2-ky^2|\leq 4k$; and infinitely many such pairs x, y can be obtained. The proof now runs on familiar lines. To demonstrate the existence of a fundamental solution the author works in the abstractly defined field of elements $a+b\sqrt{k}$. He shows at great length that this can be ordered (the order is that obtained by giving $\sqrt{k}$ its real-number value; but this interpretation is avoided). The proofs are now again the familiar ones.

J. W. S. Cassels (Cambridge, England).

Val'fiš, A. Z. Pell's equation in imaginary quadratic fields. Akad. Nauk Gruzin. SSR. Trudy Mat. Inst. Razmadze 18, 133–151 (1951). (Russian. Georgian summary)

The equation in question is $\xi^2-\kappa\eta^2=1$, where κ is a given nonsquare integer in the imaginary quadratic field $K(\sqrt{d})$, $d<0$, and ξ, η are variable integers in the field. Using the method of Dirichlet for $K(\sqrt{-1})$ [J. Reine Angew. Math. 24, 291–371 (1842) = Werke, Bd. I, Reimer, Berlin, 1889, pp. 533–618], the author shows the existence of nontrivial solutions and of a fundamental solution ξ_0, η_0 such that every other solution is of the form $\xi+\eta\sqrt{\kappa}=\pm(\xi_0+\eta_0\sqrt{\kappa})^n$ (or if $d\neq -1$ and $\kappa=-1$ or $d=-3$ and $\kappa=\frac{1}{2}(1\pm\sqrt{(-3)})$ of the form $\pm i(\xi_0+\eta_0\sqrt{\kappa})^n$). The methods are elementary throughout. There is an account of other recent work, but no reference to the relation with the units of the quartic field $K(\sqrt{d}, \sqrt{\kappa})$. *J. W. S. Cassels* (Cambridge, England).

Varnavides, P. Euclid's algorithm in real quadratic fields. Prakt. Akad. Athēnōn 24 (1949), 117–123 (1951). (Greek. English summary)

The author applies his improvement of Davenport's method [Quart. J. Math., Oxford Ser. 19, 54–58 (1948); these Rev. 9, 500] to prove the existence of Euclid's algorithm in fifteen of the seventeen real quadratic fields in which the algorithm exists. The problem of determining all quadratic fields having a Euclid algorithm was completely solved a few years ago and a brief history of this problem can be found in Hua's review of Inkeri's paper [these Rev. 10, 15]. *T. M. Apostol* (Pasadena, Calif.).

Iseki, Kanesiroo. Über die imaginär-quadratischen Zahlkörper der Klassenzahl Eins oder Zwei. Proc. Japan Acad. 27, 621–622 (1951).

Some results are announced, the proofs of which have already appeared [Jap. J. Math. 21, 145–162 (1952); these Rev. 14, 452]. *W. H. Mills* (New Haven, Conn.).

MacKenzie, Robert E. Class group relations in cyclotomic fields. Amer. J. Math. 74, 759–763 (1952).

Let F be the field obtained by adjunction of a primitive root of unity ϵ to the rational field. If q is any residue class prime to n modulo n, let σ_q be the automorphism of F which changes ϵ^q into ϵ. Let s and t be any integers; set $a(q)=[q(s+t)/n]-[sq/n]-[tq/n]$. Let $\mathfrak{R}$ be any ideal class in F. Then it is proved that $\prod_q\sigma_q(\mathfrak{R}^{a(q)})=1$. The very ingenious method of proof is as follows. Let $\mathfrak{p}$ be a prime ideal of degree 1 in F belonging to $\mathfrak{R}$; let $(\alpha/\mathfrak{p})$ be the nth power residue symbol modulo $\mathfrak{p}$. Then, for any s, $(\alpha/\mathfrak{p})^s$ is considered as a function on the additive group of residues modulo $\mathfrak{p}$ (with the convention $(0/\mathfrak{p})=0$). The author constructs the convolution of these functions relative to the integers s and t, say $g_s{}^t$. The number $g_s{}^t(1)$ is in F and its prime factorization involves only the ideal $\mathfrak{p}$ and its conjugates. By finding out the exponents in this factorization,

one obtains a relation between the class $\mathfrak{K}$ of $\mathfrak{p}$ and its conjugates which is (almost) the desired relation. Now, the problem of the determination of $g_s^{\,l}$ is equivalent to that of multiplying the Fourier transforms φ_s and φ_t of $(\alpha/\mathfrak{p})^s$ and $(\alpha/\mathfrak{p})^t$; these Fourier transforms take their values in the field E obtained from F by adjunction of the pth roots of unity. The factorisations of the values of φ_s and φ_t in prime ideals of E are determined by direct computation: it is first shown that these values involve only prime divisors of $\mathfrak{p}$ and its conjugates, and then the exponents are determined by using congruences modulo a sufficiently big ideal.

C. Chevalley (New York, N. Y.).

Hasse, Helmut. Über die Artinsche Vermutung und verwandte Dichtefragen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 116, 17 pp. (1952).

Suppose that M is the set of all primes for which a given integer a ($\neq 0, \neq \pm 1$) is a primitive root. Let k_q denote $q(q-1)$ if a is not a qth power, and $q-1$ if a is a qth power, q a prime. Then Artin conjectured that M should have the Dirichlet density $\omega(M)=\prod_q(1-1/k_q)$ by virtue of an analogue to infinite product theorems in the theory of probabilities. Now let K_q denote the normal field which is generated by $a^{1/q}$ and the qth roots of unity; furthermore, suppose that $\zeta_{a,q}(s)$ denotes the ζ-function of K_q, in which the factors $(1-N(p)^{-s})^{-1}$ for primes p of K_q dividing a or q are omitted. The author shows that Artin's conjecture will be proved if the sum $\sum_q(1/k_q)[\log \zeta_{a,q}(s)/\log \zeta(s)]$ can be shown to be uniformly convergent for $1<s\leqq s_0$, where $\zeta(s)$ is the ζ-function of the rational number field. Furthermore, the author presents H. Bilharz' proof [Math. Ann. **114**, 476–492 (1937)] (using A. Weil's proof of the Riemann Hypothesis for function fields) of the corresponding conjecture for function fields. Finally he discusses an unpublished result of H. W. Knobloch which is related to Artin's conjecture, namely, that the set of all primes p for which $p-1$ is squarefree has the density $\prod_q(1-1/q(q-1))$.

O. F. G. Schilling (Chicago, Ill.).

Hasse, Helmut. Die Einheitengruppe in einem total-reellen nicht-zyklischen kubischen Zahlkörper und im zugehörigen bikubischen Normalkörper. Miscellanea Academica Berolinensia, vol. I, pp. 1–24. Akademie-Verlag, Berlin, 1950.

Suppose that K/P is a totally real noncyclic cubic field over the rational number field with the normal field B/P having the Galois group $\mathfrak{G}=\{P\}$ and the quadratic subfield Ω/P. Let H^*, η denote the unit groups of B and Ω respectively, whereas H is the group of relative units of B/Ω, that is, of those units in H whose norms with respect to Ω equal 1. The unit groups H^* and H can be interpreted as lattices in hyperplanes E^* and E, of dimensions 5 and 4, respectively, which lie in the 6-dimensional space of logarithms $\log|A^P|$, A in the algebra $B\times\overline{\mathrm{P}}$, where $\overline{\mathrm{P}}$ denotes the field of all real numbers. In order to determine the structure of H^* it is important to know the structure of H since H^* is either the direct product of the groups η and H (the norms with respect to Ω of the units H^* are cubes) or $[\mathrm{H}^*:\eta\times\mathrm{H}]=3$ (existence of a unit $\mathrm{E}^*\,\varepsilon\, B$ whose norm with respect to Ω is a fundamental unit of Ω). Now it is noted that L, E^*, and E are representation modules of $\mathfrak{G}$, and the author then determines to which representations of $\mathfrak{G}$ in $\overline{\mathrm{P}}$ they belong. By direct computation it follows that E (to which there belongs twice the irreducible representation of degree 2) determines the same representation as the regular representation of the algebra $\mathfrak{S}\times P$ where the algebra $\mathfrak{S}$ is given over P by the relations $1+S+S^2=0$, $T^2=1$, $ST=TS^2$. In this manner a right module isomorphism with respect to $\mathfrak{S}$ is set up, and by means of it there corresponds to the lattice $\mathrm{H}\subset E$ a regular ideal $\mathfrak{C}$ of the hypercomplex order J which is determined by 1, S, T, ST over the integers. In this manner the structure of H as operator group with respect to $\mathfrak{G}$ is translated into a problem of the arithmetic structure of J. It is shown by direct computation that J is contained in precisely two distinct maximal orders which in turn are explicitly determined in terms of the elements S, T, and ST. Next, necessary and sufficient conditions stating when a regular right ideal A of J is principal in J are established by means of relative bases for A over $\mathrm{P}(S)$. In particular it follows that a nonprincipal ideal A is the set of elements of a principal ideal in a uniquely determined maximal order. As a consequence of such ideal-theoretic considerations the operator isomorphism between E and $\mathfrak{S}\times\overline{\mathrm{P}}$ determines then that either H is generated by a unit E as operator group over J ($\mathfrak{C}$ is principal in J), or that H is a group extension of order 3 over such an operator group ($\mathfrak{C}$ is not a principal ideal in J). In the latter case the author fixes the relative generator $\mathfrak{E}$ (corresponding to a well-determined element of $\mathfrak{C}$) and the relations which this element satisfies in terms of the group $\mathfrak{G}$. Furthermore, the author discusses the unit group H_0 of the cubic field K in terms of H (as invariant subgroup in the sense of the Galois theory). It turns out that, for principal ideals $\mathfrak{C}$, the units of H_0 are norms of relative units E^x, $x\,\varepsilon\, J$; if $\mathfrak{C}$ is not principal, then the norm group for E (within H_0) does depend on the particular representation of $\mathfrak{C}$ as a principal ideal in one of the imbedding maximal orders of J. Finally, the case $[\mathrm{H}^*:\eta\times\mathrm{H}]=3$ is examined and a complete survey of the defining relations for generating units is made.

O. F. G. Schilling (Chicago, Ill.).

Dufresnoy, Jacques, et Pisot, Charles. Sur un problème de M. Siegel relatif à un ensemble fermé d'entiers algébriques. C. R. Acad. Sci. Paris **235**, 1592–1593 (1952).

Dufresnoy, Jacques, et Pisot, Charles. Sur un point particulier de la solution d'un problème de M. Siegel. C. R. Acad. Sci. Paris **236**, 30–31 (1953).

Let S be the set of all algebraic integers $\theta>1$ such that all their conjugates (except θ itself) have their moduli strictly less than 1. It is known [see two papers by the reviewer: Duke Math. J. **11**, 103–108 (1944); **12**, 153–172 (1945); these Rev. **5**, 254; **6**, 206] that S is a closed set. It has been proved by C. L. Siegel [ibid. **11**, 597–602 (1944); these Rev. **6**, 39] that the two smallest numbers of S are the real roots of $z^3-z-1=0$ and of $z^4-z^3-1=0$, both being isolated points of S which contains no other point in the interval $1<\theta\leqq\sqrt{2}$. C. L. Siegel conjectured that the smallest limit point of the set is the number $\frac{1}{2}(1+\sqrt{5})$. The authors give the essential ideas of the proof which they have found of this conjecture. The whole proof, which is a long one, will appear elsewhere.

The first note contains also the following results: 1) every number $\theta\,\varepsilon\, S$ which is totally real is a limit point of S; 2) every number of the form θ^n ($\theta\,\varepsilon\, S$, $n\geqq 2$) is a limit point of S.

R. Salem (Paris).

Cohen, Eckford. Sur les congruences du deuxième degré dans les corps algébriques. C. R. Acad. Sci. Paris **235**, 1358–1360 (1952).

The author considers the congruence $\rho\equiv\alpha_1\xi_1^2+\cdots+\alpha_s\xi_s^2$ (mod A) in an algebraic number field when A is an odd

ideal and $\alpha_1, \cdots, \alpha_s$ are integral numbers with $(\alpha_i, A)=1$. He gives (without proof) the number of solutions when A is a power of a prime ideal. The general case may be reduced to this case.
H. Bergström (Gothenberg).

Carlitz, L. Primitive roots in a finite field. Trans. Amer. Math. Soc. **73**, 373–382 (1952).

According to Ore a number γ in a finite field $GF(p^n)$ belongs to the polynomial $a(x)=\sum_{r=0}^{m} a_r x^{p^r}$ $(a_m \neq 0)$ if $a(\gamma)=0$ and if there is no polynomial of this form with smaller m for which $a(\gamma)=0$. If $a(x)=x^{p^n}-x$, then γ is a primitive root in the sense of Ore; such a number need not be a primitive root in the ordinary sense. This paper investigates the question whether a number can be found in $GF(p^n)$ which is simultaneously a primitive root in both senses, and it is proved that for p^n sufficiently large such a number always exists. The generalization of this to arbitrary exponents and polynomials $a(x)$ is answered in a similar fashion. Asymptotic formulas for the number of numbers of the required kind are obtained. On the other hand, it is proved that for given p, r there exist infinitely many irreducible polynomials P so that no polynomial of degree $\leqq r$ can be a primitive root (mod P) in the sense of Ore.
H. W. Brinkmann (Swarthmore, Pa.).

Carlitz, L. A problem of Dickson. Duke Math. J. **19**, 471–474 (1952).

The author proves that if $ef=p^n-1$, p a prime, $n\geqq 1$, and if the values of the polynomial $F(x) \in GF[p^n, x]$ are eth powers of elements of the field $GF(p^n)$ for every value $x \in GF(p^n)$, then $F(x)$ is the eth power of a polynomial $H(x) \in GF[p^n, x]$, provided p^n exceeds a certain bound N_k, $k=\deg F(x)$. This includes a conjecture of Dickson for $p>2$, $k=2r>2$, which was proved earlier by the author [same J. 14, 1139–1140 (1947); these Rev. 9, 337], using the Riemann hypothesis for the Artin zeta-function proved by Weil [Proc. Nat. Acad. Sci. U. S. A. 27, 345–347 (1941); these Rev. 2, 345]. The present proof of the more general result does not employ the Riemann hypothesis, but instead some results of Davenport [Acta Math. 71, 99–121 (1939); these Rev. 1, 41]. However, the earlier method yielded a better bound for the Dickson case: $N=(k-1)^2$.
R. Hull.

Carlitz, L. Some congruences for the Bernoulli numbers. Amer. J. Math. **75**, 163–172 (1953).

Vandiver [Duke Math. J. 5, 548–551 (1939); these Rev. 1, 4] proved that $B^{a(p-1)}(B^{p-1}-1)^r \equiv 0 \pmod{p^{r-1}}$, where after expansion of the left member B_m is substituted for B^m, and $(B+1)^m=B^m$ for $m\neq 1$; p is an odd prime, and $a>0$, $r>0$, $a+r<p-1$. In the present paper this congruence is extended in several directions. The following five theorems are representative. (1) If $a\geqq 1$, $r\geqq 1$, $a+r\leqq p-1$, and $\sigma_m=(B_{m(p-1)}+p^{-1}-1)/m$, then $\sigma^a(\sigma-1)^r\equiv 0 \pmod{p^r}$. (2) If $m=(hp+k)(p-1)$, $h\geqq 0$, $k\geqq 1$, $r\geqq 1$, $k+r\leqq p-1$, then $B^m(B^{p-1}-1)^r\equiv 0 \pmod{p^{r-1}}$. (3) For $m=rp^h(p-1)$, $p\geqq 3$, the numerator of $B_m+p^{-1}-1$ is divisible by p^h, and the quotient satisfies $p^{-h}(B_m+p^{-1}-1)\equiv rw_p \pmod{p}$, $p>3$, where w_p denotes Wilson's quotient. (4) Let $(p-1)p^{e-1}|b$, $c=(p-1)u>re$; then for $r<p$ we have $B^c(B^b-1)^r\equiv 0$ (mod $p^{(r-1)e}$), except perhaps when $r=p-1$, $e=1$, when the modulus is p^{p-2}. For $r\geqq p$ define h as the least integer $\geqq(re+1)/p$; then $B^c(B^b-1)^r\equiv 0 \pmod{p^{re-h}}$. (5) Let $(p-1)p^{e-1}|b$, $r\geqq 1$; then

$$\sum_{s=0}^{r}(-1)^{r-s}C_s^r\frac{B_m+p^{-1}-1}{s+1}\equiv 0 \quad (\bmod\ p^{h-h},\ (r+1)^{-1}p^{e(r+1)-h}),$$

where $m=(s+1)b$, and $h=e$ for $r<p$, while h is the least integer $\geqq re/p$ for $r>p$.
A. L. Whiteman.

Carlitz, L. A divisibility property of the Bernoulli polynomials. Proc. Amer. Math. Soc. **3**, 604–607 (1952).

Put $xe^{xu}/(e^x-1)=\sum_{m=0}^{\infty}B_m(u)x^m/m!$, $B_m=B_m(0)$. The author has shown in an earlier paper [see the preceding review] that if $(p-1)p^r|m$, $m>0$, then $B_m+p^{-1}-1\equiv 0 \pmod{p^r}$, $p\geqq 3$; furthermore, if we put $m=t(p-1)p^r$, then

$$\sigma_m=(B_m+p^{-1}-1)/p^r\equiv tw_p \pmod{p},$$

where $w_p=((p-1)!+1)/p$. It is also stated in the earlier paper that $\sigma_m\equiv p^{1-h}\sum_{a=1}^{p^h}q(a) \pmod{p^{2h}}$, where $p>3$ and $p\nmid a$ in the summation, $h=[(r+2)/3]$ and

$$q(a)=(a^{(p-1)p^r}-1)/p^{r+1}.$$

In the present note the author first extends these results to $B_m(u)$, where the rational number u is integral (mod p). Secondly he derives the corresponding divisibility property for $B_m^{(k)}(u)$ defined by $(x/(e^x-1))^k e^{xu}=\sum_{m=0}^{\infty}B_m^{(k)}(u)x^m/m!$, $B_m^{(k)}=B_m^{(k)}(0)$, where k is restricted to the range

$$1\leqq k\leqq p-1.$$

A. L. Whiteman (Princeton, N. J.).

Carlitz, L. A note on Bernoulli numbers and polynomials of higher order. Proc. Amer. Math. Soc. **3**, 608–613 (1952).

Following the notation of Nörlund [Vorlesungen über Differenzenrechnung, Springer, Berlin, 1924] the author defines $B_m^{(k)}$, $B_m^{(k)}(u)$ by means of

$$(x/(e^x-1))^k e^{xu}=\sum_{m=0}^{\infty}B_m^{(k)}(u)x^m/m!, \quad B_m^{(k)}=B_m^{(k)}(0), \quad k\geqq 1.$$

He also puts

$$(m)_k=m(m-1)\cdots(m-k+1), \quad (m)_0=1,$$
$$[m]_k=(a^m-1)(a^{m-1}-1)\cdots(a^{m-k+1}-1), \quad [m]_0=1.$$

The following is a summary of eight theorems established by the author. In these theorems p denotes an odd prime; the rational numbers a, u are integral (mod p) and $p\nmid a$. In Theorems 2, 3, 5, and 6 it is assumed that $k<p-1$, $m\not\equiv 0, 1, \cdots, k-1 \pmod{p-1}$, $m\geqq k\geqq 1$. (1) The number $U_m^{(k)}=[m]_k B_m^{(k)}(u)/(m)_k$, $m\geqq k\geqq 1$, is integral (mod p). (2) The number $B_m^{(k)}(u)/(m)_k$ is integral (mod p); in particular $B_m^{(k)}(u)$ is integral (mod p). (3) If $p^r|(m)_k$, then the numerator of $B_m^{(k)}(u)$ is divisible by p^r. (4) If $(p-1)p^{e-1}|b$, $m\geqq rb+k, k\geqq 1$, then (*) $\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}U_{m+sb}^{(k)}\equiv 0 \pmod{p^{re}}$. (5) Put $T_m^{(k)}=B_m^{(k)}(u)/(m)_k$. If $m\geqq rb+k$, then congruence (*) holds with $U_{m+sb}^{(k)}$ replaced by $T_{m+sb}^{(k)}$. (6) If $m\geqq rb+k$, $r\geqq k$, then congruence (*) holds with $U_{m+sb}^{(k)}$ replaced by $B_{m+sb}^{(k)}(u)$ and the modulus p^{re} replaced by $p^{(r-k)e}$. (7) This theorem gives the residue of $pB_m^{(k)}(u)$ (mod p) when $k\leqq p-1$, $m\equiv s_0 \pmod{p-1}$, $0\leqq s_0\leqq k-1$. (8) This theorem gives the residue of $pB_m^{(p)}(u)$ (mod p) when $m\equiv s_0$ (mod $p-1$), $0\leqq s_0<p-1$.
A. L. Whiteman.

Selmer, Ernst S. On the Dixon elliptic functions in the equianharmonic case. Norsk Mat. Tidsskr. **34**, 105–116 (1952).

The elliptic functions $x=\mathrm{sm}\, u$, $y=\mathrm{cm}\, u$, which satisfy the equation $x^3+y^3-3\alpha xy=1$, were investigated by A. C. Dixon [Quart. J. Pure Appl. Math. 24, 167–233 (1890)]. In the present paper the writer is concerned with the special case $\alpha=0$; he writes sip u, cop u in place of sm u, cm u. These functions have been employed by Koschmieder

[Math. Ann. 83, 280–285 (1921)] to obtain a simple proof of the cubic reciprocity theorem. The author has shown that the functions possess simple trisection formulas. He now discusses in detail the behavior of the functions inside the period parallelogram and derives formulas suitable for numerical computation. *L. Carlitz* (Durham, N. C.).

★Eichler, Martin. Quadratische Formen und orthogonale Gruppen. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete. Band LXIII. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1952. xii +220 pp. 24.60 DM; Bound, 27.60 DM.

The aim of the author is to develop in this book a thoroughgoing, self-contained theory of quadratic forms from a completely modern point of view. A familiarity with modern algebra and the elements of the theory of algebraic numbers is assumed. The quadratic forms are considered as metrics of a vector space in an arbitrary field and associated with such a metric is the orthogonal group in the broad sense of an automorphic transformation of the form. The reader familiar with the classical theory will miss some landmarks: reduction theory and some details of representation theory for forms with integral coefficients. But the spirit of the work of Hasse, Hecke, and Siegel pervade this volume and the power and beauty of the general modern methods is evident throughout. The author, in consideration for the reader recommends the omission of certain sections for those who wish to avoid the deeper ramifications of the theory. This is, however, not a book for the skimming reader.

The following partial list of chapter and section titles gives an idea of the content of the book: The metric space and its automorphisms; the spinor-representation of the orthogonal group; spaces of dimension 2 to 6; fundamental properties of perfect discrete valuation fields and their quadratic extensions; invariant characteristics of spaces and space types; lattices; units; ideals; the arithmetic of Clifford algebras; Theta functions and Gauss sums; elementary theory of mass; absolute mass of a p-adic unit group; the analytic mass formula for definite spaces and general spaces; the geometric theory of units. *B. W. Jones.*

Eichler, Martin. Idealtheorie der quadratischen Formen. Abh. Math. Sem. Univ. Hamburg 18, 14–37 (1952).

The author gives a short and readable account of his work on quadratic forms (spinor representation of the "orthogonal" group of such a form; lattices belonging to quadratic forms over a field with a discrete non-archimedean valuation; ideals of lattices; the theory "in the large", i.e., over a finite algebraic field), as developed with all details in his recent book "Quadrisehe Formen und orthogonale Gruppen" [see the preceding review]. *K. Mahler* (Manchester).

Quadratische

Eichler, Martin. Die Ähnlichkeitsklassen indefiniter Gitter. Math. Z. 55, 216–252 (1952).

In the first part of this paper the author proves two theorems. Theorem 2: If S is the set of all integral rational numbers, two spinor-related lattices of an indefinite space of dimension greater than 2 over the rational number field are similar. Theorem 3: Under the hypotheses of theorem 2, two arithmetic maximal lattices of the same genus are similar. For ternary forms there is the added restriction that the reduced determinant of the lattice have no square factor greater than 4 and is not divisible by 8. These results are related to those of Meyer on the number of classes in genera of indefinite ternary quadratic forms.

The second part is concerned with results of Siegel having to do with the identity of the Zeta functions of related indefinite forms. *B. W. Jones* (Boulder, Colo.).

Mordell, L. J. The minima of some non-homogeneous functions of two variables. Duke Math. J. 19, 519–527 (1952).

The following theorem is proved for real functions $f(x,y)$ satisfying certain general geometrical conditions. For every real pair x_0, y_0 there exist x, y such that $x\equiv x_0$, $y\equiv y_0$ (mod 1) and $|f(x,y)|\leq k\max(|f(1,0)|,|f(0,1)|,|f(1,1)|,|f(1,-1)|)$, where k depends only on $f(x,y)$. Examples, where

$$X=px+qy,\quad Y=rx+sy\ (px-qr\neq 0),$$

are:

$$f(x,y)=Ye^{|X|^n}\ (1<n\leq 2),\quad k=\tfrac{1}{2};$$
$$f(x,y)=X^3+Y^3+|t|(X+Y),\quad k=\tfrac{1}{2};$$
$$f(x,y)=X|X|^{n-1}+Y|Y|^{n-1},\quad k=2^{-n}\ (n>1).$$

This last case for $n=3$ is a result due to Chalk [Proc. Cambridge Philos. Soc. 48, 392–401 (1952); these Rev. 13, 919]. The proof is geometrical and is related to that of Chalk. *L. Tornheim* (Ann Arbor, Mich.).

Cohn, Harvey. A periodic algorithm for cubic forms. Amer. J. Math. 74, 821–833 (1952).

A new algorithm is presented for the product of three real linear homogeneous forms in three integral variables, which combines some of the features of an ordinary continued fraction algorithm with some of the features of Minkowski's well-known algorithm [Gesammelte Abhandlungen, Bd. I, Teubner, Leipzig-Berlin, 1911, pp. 278–292]. Using Minkowski's work as a starting point (every lattice has a Minkowskian triple that serves as a basis), the author develops a reduction theory which fulfills the requirements that every degenerate cubic form considered is equivalent to a reduced one, that reduced forms possess neighboring forms and therefore arrange themselves into chains, that in case the three linear forms are conjugates, each spanning a totally real cubic module, the number of distinct reduced forms is finite and that the actual calculations in cubic fields are simple, as they involve only patterns of signs. *J. F. Koksma* (Amsterdam).

Swinnerton-Dyer, H. P. F. Extremal lattices of convex bodies. Proc. Cambridge Philos. Soc. 49, 161–162 (1953).

A neat proof that a critical (or, more generally, extremal) lattice of an n-dimensional convex body has at least $\frac{1}{2}n(n+1)$ points on the boundary. For $n=2$ or 3 this was proved by Minkowski [Ges. Abh., Bd. II, Teubner, Leipzig-Berlin, 1911, pp. 3–42] and for spheres and all n by Korkine and Zolotareff [Math. Ann. 11, 242–292 (1877)]; but the general result is apparently novel. *J. W. S. Cassels* (Cambridge, England).

Hlawka, Edmund. Zur Theorie des Figurengitters. Math. Ann. 125, 183–207 (1952).

In this memoir many of the known results of the geometry of numbers, including theorems of Minkowski, Blichfeldt, Siegel, Khintchine, and Koksma, are extended to more general situations. The most general situation contemplated is that in which the n-dimensional Euclidean space is replaced by a locally compact space R having a measure theory and satisfying certain other conditions, and a lattice

is replaced by a discontinuous group Γ of measure-preserving automorphisms of R, with a bounded and measurable fundamental region. By a "Figurengitter" the author means the system of sets γA obtained from a given measurable set A in R by the application of all automorphisms γ of Γ. The various extensions of the classical theorems require the imposition of various additional restrictions on R and Γ and on the functions introduced. Some new results for the classical case are also given. One is the following [(28) of §4]: let $f(\mathfrak{x})\leqq 1$ define a convex body of volume V in n-dimensional space, symmetrical about 0, and let $M_1, \cdots, M_n$ denote the successive Minkowski minima of $f(\mathfrak{x})$ for a lattice of determinant Δ. Then for every point $\mathfrak{x}_0$ of space there is a lattice point $\mathfrak{x}$ such that

$$f(\mathfrak{x}-\mathfrak{x}_0)\leqq \tfrac{1}{2}M_n\left\{\frac{2^n\Delta}{VM_1\cdots M_n}\right\},$$

where $\{t\}$ denotes the least integer $\geqq t$. *H. Davenport.*

Scherk, Peter. Convex bodies off center. Arch. Math. **3**, 303 (1952).

Let $F(\mathfrak{x})$ be a distance function in n-space such that $F(\mathfrak{x})\leqq 1$ defines a symmetric convex body of volume J. Suppose that $\mathfrak{x}=\mathfrak{o}$ is the only integral vector satisfying $F(\mathfrak{x})\leqq 2t\,J^{-1/n}$. Then it is shown that to every vector $\mathfrak{a}$ there is an integral vector $\mathfrak{g}$ such that

$$F(\mathfrak{a}-\mathfrak{g})<\frac{1+(n-1)t^n}{t^{n-1}J^{1/n}}.$$

This improves further a result due to Mahler recently improved by H. Störmer and G. Walter [Arch. Math. **2**, 346–348 (1949); these Rev. **13**, 115]. A slightly stronger form of the author's result is contained in the paper of E. Hlawka reviewed above. *C. A. Rogers* (London).

Chabauty, Claude. Empilement de sphères égales dans R^n et valeur asymptotique de la constante γ_n d'Hermite. C. R. Acad. Sci. Paris **235**, 529–532 (1952).

The object of this paper is to show that the Hermite constant γ_n associated with positive definite quadratic forms is asymptotically equal to $n/(2\pi e)$; this result would supersede the known inequalities

$$\frac{1}{2\pi e}\leqq\liminf n^{-1}\gamma_n\leqq\limsup n^{-1}\gamma_n\leqq\frac{1}{\pi e}.$$

Pressure of space has forced the author to condense his argument and omit certain steps in the proofs of his lemmas, and for this reason the paper is hard to read. The reviewer has been in correspondence with the author, who has informed him that it is not always possible, as stated in support of Lemma 5, to choose points A, B from a set of $n+2$ points on the unit sphere in R^n so that the segment AB meets the convex cover of the remaining n points. Until this part of the argument can be amended, the main result of the paper must remain in doubt. A correction will be published in due course. *R. A. Rankin* (Birmingham).

Chabauty, Claude. Nouveaux résultats de géométrie des nombres. C. R. Acad. Sci. Paris **235**, 567–569 (1952).

Utilisant les définitions et les notations d'une note antérieure [voir la note analysée ci-dessus] l'auteur démontre le théorème suivant: Si $G\subset R^n$, pour tous les entiers naturels h et k, on a

$$((n+2)^{kh}-1)D(G)\geqq\Omega_n\rho^n(h, G)((k-1)(n+1)k^{-1}n^{-1})^{n/2}$$

et en particulier

$$D^{2/n}(G)\geqq 4\pi en^{-1}\rho^2(h, G)(1-\epsilon(nk^{-1})),$$

où $\epsilon(t)$ désigne une fonction positive qui$\to 0$ avec t^{-1}.

Les applications de ce théorème au cas d'un système de formes linéaires donnent des améliorations des résultats classiques de Minkowski et des résultats de Blichfeldt [Monatsh. Math. Phys. **43**, 410–414 (1936); **48**, 531–533 (1939); ces Rev. **1**, 68], Hlawka [Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **156**, 247–254 (1948); ces Rev. **10**, 236], Rankin [Indagationes Math. **10**, 274–281 (1948); ces Rev. **10**, 284], et Rogers [Acta Math. **82**, 185–208 (1950); ces Rev. **11**, 501]. *J. F. Koksma.*

Spiegel, M. R. On a class of irrational numbers. Amer. Math. Monthly **60**, 27–28 (1953).

Let $a_1, a_2, \cdots$ be an infinite sequence of integers, infinitely many of which are different from 0. Put

$$\phi=\sum_{n=1}^{\infty}\frac{a_n}{r^n(n!)^b}.$$

Assume that for all sufficiently large n, $|a_n|<Cn^\alpha$ where C and α are constants, $\alpha<b$. The author proves that ϕ is irrational. *P. Erdös* (Los Angeles, Calif.).

Redheffer, R. M. Power series and algebraic numbers. Amer. Math. Monthly **60**, 25–27 (1953).

The author proves the following theorem: Let $f(z)=\sum a_kz^k$ be a rational function with algebraic coefficients; assume further that $f(z)$ is not a polynomial and that $f(z)$ is regular at 0. Let m be the multiplicity of that root of the denominator of $f(z)$ which has maximum multiplicity. Then, apart from a set of at most m values, the value of the function $g(z)=\sum a_kz^k/k!$ is transcendental when the value of z is algebraic. Some related questions are discussed. *P. Erdös* (Los Angeles, Calif.).

ANALYSIS

Blumenthal, Leonard M. Two existence theorems for systems of linear inequalities. Pacific J. Math. **2**, 523–530 (1952).

The writer develops the theory of linear inequalities by means of metric methods which he initiated in a previous paper [Duke Math. J. **15**, 955–966 (1948); these Rev. **10**, 470]. Preliminaries—E_n: n-dimensional euclidean vector space. $p_1, p_2, \cdots, p_n, p_{n+1}$: vectors of E_n such that $p_1, p_2, \cdots, p_{n-1}$ are linearly independent. L: linear manifold spanned by $p_1, \cdots, p_{n-1}$. x_{il} $(l=1, \cdots, n)$: components of p_i $(i=1, \cdots, n+1)$ with respect to a normed orthogonal basis. p_n and p_{n+1} lie on the same or on opposite sides of L according as the product of the signed volumes

$$\begin{aligned}V(p_1, \cdots, p_{n-1}, p_n)\cdot V(p_1, \cdots, p_{n-1}, p_{n+1})&=|x_{kl}|\cdot|x_{hl}|\\ (k, l=1, 2, \cdots, n,\ h=1, 2, \cdots, n-1, n+1)&\\ &=|\text{sum } x_{kl}x_{hl}|=|(p_k, p_h)|\end{aligned}$$

is positive or negative. If the p_i's are unit vectors, $(p_k, p_h)=\cos p_kp_h$. Theorems—System $(\mathcal{S}+)$ of linear inequalities in n indeterminates with rank $r+1$: $(p_i, x)\geqq 0$, $i=1, \cdots, m$, one at least of the left members having to be positive. $A=\|p_i\|$, $A^T=$transpose of A, $B=\|AA^T\|$. The

system ($\mathcal{I}+$) has a solution if and only if a shifting of rows and corresponding columns exists such that (i) the upper left principal minor M of B of order $r+1$ does not vanish, (ii) each minor of B formed from M by replacing its last row with that part of the jth row of B contained in the first $r+1$ columns ($j=r+2, r+3, \cdots, m$) is positive or zero. The second theorem is concerned with the existence of a nontrivial solution of the same system of inequalities, the positivity requirement being dropped. *Chr. Pauc.*

Tagamlickiĭ, Ya. A. On the Newton interpolation series with non-negative coefficients. Doklady Akad. Nauk SSSR (N.S.) 87, 183–186 (1952). (Russian)

Let $F(x)=(x-\alpha_1)\cdots(x-\alpha_n)$ and define

$$[f(t), \alpha_1, \cdots, \alpha_n]=f(\alpha_1)/F'(\alpha_1)+\cdots+f(\alpha_n)/F'(\alpha_n).$$

Let $0<x_n\uparrow L\leqq\infty$, $\sum 1/x_n=\infty$. The author proves that a function of class C^∞ in $0<x<L$ can be developed in a Newton series $a_0+\sum_{k=1}^\infty\prod_{j=1}^k(x_j-x)$ with all $a_k\geqq 0$ if and only if

$$(-1)^n[f(t), x, x_1, \cdots, x_n]\geqq 0,$$
$$(-1)^{n+1}[f(t), x, \xi, x_1, \cdots, x_n]\geqq 0,$$

$0<x<x_{n+1}$, $0<\xi<x_{n+1}$, for all n. *R. P. Boas, Jr.*

Berman, D. L. Solution of an extremal problem of the theory of interpolation. Doklady Akad. Nauk SSSR (N.S.) 87, 167–170 (1952). (Russian)

Let $\{x_j^{(n)}\}$ be a set of interpolation points in $[-1, 1]$ and let $l_j^{(n)}(x)$ be the corresponding Lagrange interpolating polynomials. The author seeks the minimum over all sets of interpolation points of the maximum over x of $\sum_{j=0}^n|[l_j^{(n)}(x)]^{(k)}|$, and finds that it is equal to $T_n^{(k)}(1)$, where $T_n(x)=\cos\{n\cos^{-1}x\}$, and is attained for $x_j^{(n)}=\cos(j\pi/n)$. The proof depends on the result [Duffin and Schaeffer, Trans. Amer. Math. Soc. 50, 517–528 (1941); these Rev. 3, 235] that $|P(\cos j\pi/n)|\leqq 1$ implies $|P^{(k)}(x)|\leqq T_n^{(k)}(1)$ if $P(x)$ is a polynomial of degree at most n. The corresponding problem is solved when $[l_j^{(n)}(x)]^{(k)}$ is replaced by $[l_j^{(n)}(x+iy)]^{(k)}$; the extremal value is $|T_n^{(k)}(1+iy)|$.

R. P. Boas, Jr. (Evanston, Ill.).

Gagua, M. E. On the approximation of continuous functions by special solutions of elliptic differential equations. Soobščeniya Akad. Nauk Gruzin. SSR. 11, 211–214 (1950). (Russian)

Let E be a closed bounded set in the (x, y)-plane such that for each integer n there is a region T_n of connectivity $m+1$ with $T_n\subset E$ and meas $T_n<1/n$. It is proved that, if $f(x, y)$ is continuous on E, it may be approximated uniformly by linear combinations of fixed sets $h_k(x, y; x_0, y_0)$, $w_k(x, y; x_i, y_i)$, $k=0, 1, \cdots, i=0, 1, \cdots, m$, of particular solutions of the differential equation

$$\Delta u+a(x, y)u_x+b(x, y)u_y+c(x, y)u=0.$$

The coefficients a, b, c are assumed to be entire functions of x and y while the h_k are entire functions of x and y and the w_k are regular in x and y except for $(x, y)=(x_i, y_i)$.

P. Davis (Washington, D. C.).

Gagua, M. On the best approximation of solutions of differential equations of elliptic type. Doklady Akad. Nauk SSSR (N.S.) 86, 7–10 (1952). (Russian)

The author generalizes the problem of the best approximation of harmonic functions by harmonic polynomials which has been studied by Walsh, Sewell, and Elliott, [Trans. Amer. Math. Soc. 67, 381–420 (1949); these Rev. 11, 515] and by Walsh and Elliott [ibid. 68, 183–203 (1950); these Rev. 11, 515]. Central to the development is an operator

$$(1)\quad u(x, y)=P(\phi)=G(z, 0, z, \bar z)\phi(z)-\int_0^z\phi(t)\frac{\partial}{\partial t}G(t, 0, z, \bar z)dt,\quad z=x+iy,\quad \bar z=x-iy,$$

which maps analytic functions of a complex variable $\phi(z)$ into solutions u of the differential equation

$$(2)\qquad \Delta u+a(x, y)u_x+b(x, y)u_y+c(x, y)u=0.$$

The coefficients in (2) are assumed to be entire and G is the complex Riemann function for (2). (This operator is closely related to S. Bergman's integral operator of the first kind.) The generalizations of harmonic polynomials are taken to be the particular solutions

$$(3)\qquad u_0=G(0, 0, z, \bar z),\quad u_k=\operatorname{Re}P(z^k),\quad v_k=\operatorname{Im}P(z^k).$$

The set of solutions of the form $w_n=\sum_{i=0}^n a_iu_i+\sum_{i=1}^n b_iv_i$ is designated by M_n, and for an arbitrary solution u in a bounded simply connected region T set

$$(4)\qquad E_n(u, T)=\inf_{w_n\in M_n}\{\sup_{(x,y)\in\bar T}|u(x, y)-w_n(x, y)|\}.$$

The author obtains a number of theorems which relate the asymptotic behavior of $E_n(u)$ to the differentiability properties of U on the boundary and on the exterior level lines of T. Thus, e.g., let $E_n(u, T)\leqq M_n-k-\alpha M>0$, $0<\alpha\leqq 1$, k integer. Then u is a regular solution of (2) which has boundary values which possess a kth derivative satisfying a Hölder condition of index α if $0<\alpha<1$, and if $\alpha=1$, satisfying

$$|u^{(k)}(s_1)-u^{(k)}(s_2)|\leqq\text{const.}\times|s_1-s_2|\cdot|\log|s_1-s_2||.$$

P. Davis (Washington, D. C.).

Gagua, M. On estimates of the best approximation of solutions of certain differential equations of elliptic type. Doklady Akad. Nauk SSSR (N.S.) 86, 225–228 (1952). (Russian)

The author continues the studies described in the previous review. In particular, inequalities are derived between $E_n(u, T)$ and the modulus of continuity of u in $\bar T$.

P. Davis (Washington, D. C.).

★Mandelbrojt, S. Séries adhérentes, régularisation des suites, applications. Gauthier-Villars, Paris, 1952. xiv+277 pp. 4000 francs.

This is a detailed exposition of work done during the past few years, mostly by the author and his students, and centering around his "fundamental inequality." [For a short account of the method and some of the applications see the author's lecture, Bull. Amer. Math. Soc. 54, 239–248 (1948); these Rev. 9, 416.] The majority of the results appear here in still more refined forms than those given previously. However, since the results are for the most part quite complicated technically, being rather delicate refinements of more familiar and more primitive theorems, it is not possible to give in a short review more than brief indications of the kind of theorems which are proved.

The first chapter deals with the regularization of sequences, in a more general form than in previous studies by the author [La régularisation des fonctions, Actualités Sci. Ind., no. 733, Hermann, Paris, 1938; Rice Inst. Pamphlet 29, no. 1 (1942); these Rev. 3, 292]. The second chap-

ter takes up some preliminary material on functions which are regular in a curvilinear strip, in particular the form of Ahlfors's distortion theorem which the author needs and various versions of the generalized problem of Watson. The latter is the problem of finding conditions on $G(x)$ and $\{M_n\}$ such that when $F(z)$ is regular for $x>x_0$, $|y|<G(x)$ and $|F(z)|\leqq M_n e^{-nz}$, it necessarily follows that $F(z)\equiv 0$.

The fundamental chapter is Chapter 3, and the fundamental theorem is to the effect that if a function is regular in a curvilinear strip, and represented there in a generalized asymptotic sense by a Dirichlet series (usually divergent) with given exponents, the coefficients of the Dirichlet series admit certain estimates (involving all the data). Numerous variants are discussed. The relation among the shape of the strip, the distribution of the exponents, and the form of the asymptotic representation is what the author means by a relation of adherence; the phrase in the first part of the title is, as the author points out in the preface, never used in the book.

Chapter 4 deals with quasi-analyticity, first in the classical sense and then in a generalized sense. The author gives the classical theorem of Carleman, first with the real-variable proofs due to Bang and himself, then with a complex-variable proof which leads to generalizations. The generalized problem is to connect the numbers $\{M_n\}$, determining a class of functions satisfying $|f^{(n)}(x)|\leqq k^n M_n$, with a sequence of integers $\{\nu_n\}$ in such a way that if $f^{(\nu_n)}(a)=0$ and $f(x)$ belongs to the class determined by $\{M_n\}$, then $f(x)\equiv 0$; the problem is significant only on an infinite interval. The direct theorems are complemented by inverse theorems showing that the results can hardly be sharpened much further. There are also theorems which the author calls "composition theorems" from their analogy with theorems on the composition of singularities of power series. The typical theorem deals with two sequences $\{M_n\}$, $\{M_n'\}$ and the complementary sequences $\{\nu_n\}$, $\{\lambda_n\}$ of integers, and says that (under appropriate hypotheses) if the pair $\{M_n\}$, $\{\nu_n\}$ does not have the generalized quasi-analytic property, the pair $\{M_n'\}$, $\{\lambda_n\}$ does.

Chapter 5 is concerned with various problems of uniqueness: closure of a sequence $\{x^{\nu_n}/F(x)\}$; uniqueness of the generalized Stieltjes and Hamburger moment problems (generalized by having x^{ν_n} instead of x^n in the definition of the moments); determination of the values at a point of the iterates of the kernel of an integral equation by means of the values of a sequence of the iterates.

Chapter 6 deals with classes of functions defined by inequalities satisfied by their derivatives, and in particular with inclusion and equivalence relations among such classes. This chapter is connected with the rest of the book by the facts that quasi-analytic classes are classes of the kind under consideration and that various kinds of regularization play a decisive role; the results of Chapter 3 are not used here.

The seventh chapter is devoted to theorems on the analytic continuation of Dirichlet series and, in particular, of power series with gaps. The emphasis is less on the location of singular points than on theorems on the order of a Dirichlet series in a strip and on theorems of the Picard type for a strip.

No one who is interested in any of the topics discussed in this book can afford not to read it. Although many of the results are given in what seems to be a definitive form, it is clear that further applications of the same techniques to other problems will be fruitful. *R. P. Boas, Jr.*

San Juan, R. Summation of divergent series and best asymptotic approximation. Mem. Real Acad. Ci. Madrid. Ser. Ci. Exact. 2, 112 pp. (1942). (Spanish)

This memoir was awarded a prize by the Madrid Academy in 1935. It contains, among other things, a proof of a proposition on asymptotic series equivalent to the theorem on quasi-analytic classes recently published by the author [C. R. Acad. Sci. Paris **238**, 118–119 (1952); these Rev. **13**, 932], and attributed by the reviewer to T. Bang [1946]. Parts of the contents of the memoir under review were reproduced in another paper [Acta Math. **75**, 247–254 (1943); these Rev. **7**, 8], where the author's notion of "optimum asymptotic approximation" is clearly explained. The author's central idea is to define the sum of a divergent series $\sum a_n$ as the limit as $z\to 1$ of the function $f(z)$ attached to $\sum a_n z^n$ in the sense of asymptotic representation (even when the power series diverges). He shows that the Stieltjes moment method fits this framework: in this method, if $(-1)^n a_n$ are Stieltjes moments $\int_0^\infty t^n\alpha(t)dt$, $\alpha(t)\geqq 0$, the sum of $\sum a_n z^n$ is taken to be (*) $\int_0^\infty(1+tz)^{-1}\alpha(t)dt$. The author gives a detailed investigation of the connection between the Stieltjes moment problem (with $\alpha(t)$ no longer required to be nonnegative) and the integrals $\int_0^\infty \alpha(t)E_p(tz)dt$, where $E_p(z)$ is the Mittag-Leffler function $\sum z^n/(np)!$. He proves that μ_n are Stieltjes moments if (for some $p>0$) $\mu_n=(np)!g(n)$, where $g(z)$ is regular for $\Re(z)>-\gamma$, $\gamma>1$, and $|g(z)|=O(e^{-\theta|z|})$, $0<\theta<p\pi/2$. There is also a study of the discrete moment problem $\mu_n=\sum_{t=0}^\infty a_t t^{(n}$, where $t^{(n}=t(t-1)\cdots(t-n+1)$, and of the corresponding summation method, with applications to the Euler summation method. A number of properties of functions of the form (*) are investigated: quasianalyticity; expression as a Laplace transform, as a factorial series, as a series of polynomials. In addition there is a large amount of expository material. *R. P. Boas, Jr.* (Evanston, Ill.).

Theory of Sets, Theory of Functions of Real Variables

Gillman, Leonard. On intervals of ordered sets. Ann. of Math. (2) 56, 440–459 (1952).

It is shown that the proposition $Q(\aleph_\alpha)$: "Every ordered set of power $\aleph_\alpha$ has a set of $\aleph_\alpha$ mutually exclusive intervals", is false if $\aleph_\alpha$ is regular and not strongly inaccessible, or if $\aleph_\alpha$ is a power of 2, but is true for some singular cardinal numbers $\aleph_\alpha$. $Q(\aleph_\alpha)$ is true for every singular $\aleph_\alpha$ if and only if to every ordinal number β there corresponds a natural number n such that $2^{\aleph_\beta}=\aleph_{\beta+n}$. If $\aleph_\alpha$ is strongly inaccessible, then every ordered set of power greater than $\aleph_\alpha$ has a set of $\aleph_\alpha$ mutually exclusive intervals, but it is not known whether $Q(\aleph_\alpha)$ is true or false. In addition to some further related results, the author derives several necessary and sufficient conditions for a nonenumerable inaccessible ω_α to be a ρ_0-number [defined by Mahlo, Ber. Verh. Sächs. Ges. Wiss. Leipzig. Math.-Phys. Kl. 63, 187–225 (1911), p. 196] and proves some miscellaneous theorems on decompositions of sets, of which the following is an instance: Let α be an arbitrary ordinal number, M be an $\aleph_\alpha$-homogeneous ordered set (i.e., M, as well as each of its intervals, is of power $\aleph_\alpha$), and $\{S_\beta\}$, $\beta<\omega_\alpha$, be a set of mutually exclusive $\aleph_\alpha$-homogeneous subsets of M whose union is M and each of which is dense in M. Then there exists a set $\{T_\gamma\}$, $\gamma<\omega_\alpha$, with the aforesaid properties of $\{S_\beta\}$, $\beta<\omega_\alpha$, and such that $|S_\beta\cap T_\gamma|=1$ $(\beta<\omega_\alpha, \gamma<\omega_\alpha)$. *F. Bagemihl.*

Sierpiński, W. Coup d'oeil sur l'état actuel de l'hypothèse du continu. Elemente der Math. 8, 1–4 (1953).

Jones, F. Burton. On the separation of the set of pairs of a set. J. Elisha Mitchell Sci. Soc. 68, 44–45 (1952).

The author proves the following theorems. Let ϕ be an infinite set, split the pairs of ϕ into two classes; then there exists an infinite subset ϕ' of ϕ all of whose pairs are in the same class. Further, if ϕ has power $\aleph_1$, the author gives a splitting of the pairs of ϕ into two classes so that if ϕ' is any subset of ϕ of power $\aleph_1$, the pairs of ϕ' are never in the same class.

The author overlooked the fact that his theorem is not new [see F. P. Ramsay, Proc. London Math. Soc. (2) 30, 264–286 (1930)]; his example of the above splitting of the pairs of a set of power $\aleph_1$ has been given previously by Sierpinski. *P. Erdös* (Los Angeles, Calif.).

Rothberger, Fritz. On the property C and a problem of Hausdorff. Canadian J. Math. 4, 111–116 (1952).

The author continues his investigation of interconnection of certain problems involving $\aleph_1$ and Ω [see, for example, Fund. Math. 35, 29–46 (1948); these Rev. 10, 689]. The unifying link in these problems seems to be that of existence of Ω-limits [Hausdorff, ibid. 26, 241–255 (1936)] defined as follows with regard to the family of all sets of natural members. For two such sets the order relation $A<B$ is to mean that $A-B$ is finite and $B-A$ infinite; a transfinite ascending sequence of sets A_μ $(\mu<\Omega)$ is said to have A as Ω-limit if no set can be intercalated between all A_μ and A. The principal (or, anyway, the strongest) result in this paper concerns a problem raised by Sierpinski [ibid. 1, 224 (1920), problem 6]. It is shown that if Ω-limits fail to exist, then the union of any $\aleph_1$ sets of real numbers of first category is also of first category. Thus nonexistence of Ω-limits implies $\aleph_1<2^{\aleph_0}$; in fact, it implies more, namely $2^{\aleph_1}=2^{\aleph_0}$, as the author has previously shown. *W. Gustin* (Bloomington, Ind.).

Green, John W. On families of sets closed with respect to products, translations, and point reflections. Anais Acad. Brasil. Ci. 24, 241–244 (1952).

A family F of subsets of E^n is called a PT-family provided (i) the intersection of any collection of sets of F belongs to F; (ii) whenever $Y\in F$ and $x\in E^n$, then $x+Y\in F$ and $2x-Y\in F$. The author establishes several properties of PT-families, among which are the following. (1) Every non-null PT-family includes a subgroup of E^n as one of its members. (2) If a PT-family includes a proper subset of E^n, then it includes a set of interior measure zero. [The author has informed the reviewer that the question raised in the last paragraph should end ". . . non-degenerate convex set?".] *V. L. Klee, Jr.* (Charlottesville, Va.).

Behrend, F. A. Zum Metrisierbarkeitsbegriff von K. Wagner. Math. Ann. 125, 140–144 (1952).

K. Wagner [Math. Ann. 120, 502–513 (1949); these Rev. 10, 518] defined properties of a "generalized metric" defined in an ordered set M (with first element 0) with values in M. In the present paper it is shown that Wagner's concept can be reformulated in terms of an addition in M, which becomes "half" of an ordered abelian group precisely in the presence of Wagner's axiom of "einfach." This reformulation permits simpler proofs and extensions of Wagner's uniqueness theorems, partly through the applicability of results of Hölder and G. Birkhoff. *R. Arens.*

Mendelsohn, N. S. Representations of positive real numbers by infinite sequences of integers. Trans. Roy. Soc. Canada. Sect. III. (3) 46, 45–55 (1952).

Let α_n, $-\infty<n<\infty$, be a sequence of real numbers, $\alpha_0=1$, $\alpha_k<\alpha_{k+1}$ for every k, $\alpha_n\to\infty$, $\alpha_{-n}\to 0$ as $n\to\infty$. Further let λ_n, $-\infty<n<\infty$, be a sequence of positive integers. The author proves the following theorem: A necessary and sufficient condition that every real positive number N be expressible uniquely in the form $N=\sum_{i=r}^{\infty}a_i\alpha_i$ $(-\infty<r\leqq i)$, where $0\leqq a_i\leqq\lambda_i$ and, for infinitely many i, $a_i<\lambda_i$, is that for all k, $-\infty<k<\infty$, $\alpha_{k+1}=\sum_{i=-\infty}^{k}\lambda_i\alpha_i$. This result generalizes the decimal representation of real numbers and also results of Faber and Cantor. The author discusses several related results. *P. Erdös* (Los Angeles, Calif.).

Conforto, Fabio. Alcune considerazioni sui numeri reali. Archimede 4, 133–142 (1952).

Expository paper on the introduction of real numbers and p-adic numbers.

Enomoto, Shizu. On completely additive classes of sets with respect to Carathéodory's outer measure. Proc. Japan Acad. 27, 627–631 (1951).

This paper is somewhat related to a previous paper of the author [same Proc. 27, 208–213 (1951); these Rev. 13, 729]. Let μ be a Carathéodory outer measure defined on an abstract space X. Designate by $\mathfrak{C}(\mu)$ the class of all μ-measurable sets. It is assumed that X is the union of a monotone increasing sequence of measurable sets K_n with finite $\mu(K_n)$. A system $\mathfrak{M}$ of sets is said to be completely additive if with $A_i\in\mathfrak{M}$ also $S_{i=1}^{\infty}A_i\in\mathfrak{M}$ and if with $A\in\mathfrak{M}$ also the complement $CA\in\mathfrak{M}$. The author calls $\mathfrak{M}$ "μ-completely additive" (μ-c.a.) when $\mathfrak{M}$ is completely additive and $\mu(S_{i=1}^{\infty}A_i\cdot K_n)=\sum_{i=1}^{\infty}\mu(A_i\cdot K_n)$ (for all n) holds if the A_i are disjoint sets of $\mathfrak{M}$. Analogously, finitely additive and μ-finitely additive (μ-f.a.) are defined. $\mathfrak{R}(\mu)$ designates the class of all sets A such that $\mu(K_n\cdot A)=\mu(K_n)-\mu(K_n\cdot CA)$ for all n; and $\mathfrak{C}(\mu)$ denotes the class of all sets E such that $\mu(A)=\mu(A\cdot E)+\mu(A\cdot CE)$ for all $A\in\mathfrak{R}(\mu)$ (with $\mu(A)<+\infty$). The main results proved by the author are the following. (a) The class $\mathfrak{C}(\mu)$ is μ-c.a. (b) If $\mathfrak{M}$ is μ-f.a., then the smallest completely additive class containing $\mathfrak{M}$ is μ-c.a. (c) Let $\mathfrak{M}$ and $\mathfrak{A}$ be μ-c.a. classes. In order that the smallest completely additive class over $\mathfrak{M}$ and $\mathfrak{A}$ be μ-c.a., it is necessary and sufficient that

$$\mu(B\cdot K_n)=\mu(B\cdot M\cdot K_n)+\mu(B\cdot CM\cdot K_n)$$

holds for all $B\in\mathfrak{A}$, $M\in\mathfrak{M}$. (d) Let $\mathbf{M}=\{\mathfrak{M}_\alpha\}$ be the system of all μ-c.a. classes of sets. Then $\mathbf{M}$ is an ordered system if $\mathfrak{M}_\alpha\leqq\mathfrak{M}_\beta$ means that $\mathfrak{M}_\alpha\subset\mathfrak{M}_\beta$. For each $\mathfrak{M}\in\mathbf{M}$ there exists a maximal element $\mathfrak{M}^*$ of $\mathbf{M}$ such that $\mathfrak{M}\leqq\mathfrak{M}^*$. (e) The union and the intersection of all maximal μ-c.a. classes $\mathfrak{M}_\alpha^*\in\mathbf{M}$ coincide with $\mathfrak{R}(\mu)$ and with $\mathfrak{C}(\mu)$, respectively. *A. Rosenthal* (Lafayette, Ind.).

Haupt, Otto, und Pauc, Christian Y. Vitalische Systeme in Booleschen σ-Verbänden. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1950, 187–207 (1951).

In the theory of differentiation of set functions, Vitali systems (V. systems) are used in an essential manner. The authors now study these V. systems in the very general case of Boolean σ-algebras. Let $\mathfrak{g}$ and $\mathfrak{z}$ be two Boolean σ-algebras with common unity E such that $\mathfrak{g}$ is an extension of $\mathfrak{z}$. The nulsoma of $\mathfrak{g}$ is designated by θ. Let $\mu|\mathfrak{z}$ be a measure which is complete in $\mathfrak{g}$ and such that every $A\in\mathfrak{z}$ is the union of countably many $A_\nu\in\mathfrak{z}$ with $\mu(A_\nu)<+\infty$; let $\mathfrak{n}$ be the

σ-ideal of the μ-nulsomas; and let $\bar{\mu}|\mathfrak{g}$ be the outer measure determined in $\mathfrak{g}$ by $\mu|\mathfrak{z}$. Any two somas A, $B \in \mathfrak{g}$ are called θ-disjoint or $\mathfrak{n}$-disjoint if $A \cap B = \theta$ or $= \theta \pmod{\mathfrak{n}}$, respectively. Notation used for both cases: A, B are $\mathfrak{n}'$-disjoint (with $\mathfrak{n}' = (\theta)$ or $\mathfrak{n}' = \mathfrak{n}$). Let $\mathfrak{c}$ be a partial system of $\mathfrak{z}$, containing θ and such that $\mu(B) < +\infty$ for every $B \in \mathfrak{c}$. Then the system $\mathfrak{c}$ is called a V. system (with respect to $X \in \mathfrak{g}$) if to every $A \in X|\mathfrak{g}$ and to every $\epsilon > 0$ there exist denumerably many $C_\rho \in \mathfrak{c}$ $(\rho = 1, 2, \cdots)$ whose union is called S, such that $A \subseteq S \pmod{\mathfrak{n}}$, $\mu(S - \tilde{A}) < \epsilon$ (where $\tilde{A}$ is a measure-cover of A) and either (1) the C_ρ are $\mathfrak{n}'$-disjoint or (2) $\sum_{\rho=1}^{r} \mu(C_\rho) - \mu(\bigcup_{\rho=1}^{r} C_\rho) < \epsilon$ for every r. In case (1) the system $\mathfrak{c}$ is called an $\mathfrak{n}'$-strong V. system, in the case (2) $\mathfrak{c}$ is called a weak V. system.

Sufficient conditions for an $\mathfrak{n}'$-strong or a weak V. system are formulated, but the proofs will be published at another place. These sufficient conditions consist, besides an approximation property, of the following "strong" or "weak halo property", respectively. (a) "Strong halo property" of $\mathfrak{c}$ $(\mathrm{mod}\ \mathfrak{n}')$: There exists a finite number $\beta = \beta(\mathfrak{c}) > 0$ such that for every $C \in \mathfrak{c}$ and for arbitrarily, but finitely, many $C_\nu \in \mathfrak{c}$ with $C_\nu \cap C \neq \theta$ $(\mathrm{mod}\ \mathfrak{n}')$ and $\mu(C_\nu) \leqq \mu(C)$ we have $\mu(\bigcup_\nu C_\nu) \leqq \beta\mu(C)$. (b) "Weak halo property" of $\mathfrak{c}$: To every α with $0 < \alpha < 1$ there exists a finite $\beta = \beta(\mathfrak{c}, \alpha) > 0$ such that for every F which is the union of finitely many $C \in \mathfrak{c}$ and for arbitrarily, but finitely, many $C_\nu \in \mathfrak{c}$ with $\mu(C_\nu) \leqq \alpha^{-1}\mu(C_\nu \cap F)$ we have $\mu(\bigcup_\nu C_\nu) \leqq \beta\mu(F)$. It has to be remarked that (a) is not a special case of (b).

Then it is proved that the existence of weak V. systems in two factors implies the existence of such systems in their product. It is also proved that the weak halo property in two factors implies the same property in the product, provided that at least one factor $\mathfrak{c}$ has the "strengthened" weak halo property, i.e., the weak halo property must be satisfied for all $F \in \mathfrak{c}_0$ where $\mathfrak{c}_0$ is the smallest Boolean algebra over $\mathfrak{c}$ (in $\mathfrak{g}$). *A. Rosenthal* (Lafayette, Ind.).

Pauc, Christian. Les théorèmes fort et faible de Vitali et les conditions d'évanescence de halos. C. R. Acad. Sci. Paris **232**, 1727–1729 (1951).

For the general case of Boolean σ-algebras some definitions are formulated and two theorems are stated, but without any proofs. Theorem I, generalizing a theorem of de Possel [J. Math. Pures Appl. (9) **15**, 391–409 (1936), p. 405], states that the weak Vitali property [see the preceding review] is equivalent to each of two other properties. Theorem II states that the weak Vitali property is implied by two other properties. As a consequence of Theorem II, a result of C. A. Hayes and A. P. Morse [Proc. Amer. Math. Soc. **1**, 107–126 (1950), Theorem 4.3; these Rev. **11**, 425] can be interpreted in the following way: If an "annular blanket" is a weak Vitali system, then it is also a strong one. Another application gives the result that the spheres of a Finsler space F form a strong Vitali system in F.

A. Rosenthal (Lafayette, Ind.).

Mařík, Jan. The Lebesgue integral in abstract spaces. Časopis Pěst. Mat. **76**, 175–194 (1951). (Czech)

This paper, largely expository in nature, deals with Daniell's extension of a linear functional J on a linear space Z of real-valued functions defined on an abstract set M. As usual, it is postulated that $z \in Z$ and $c \geqq 0$ imply min $(z, c) \in Z$, and that $z_n \in Z$ and $z_n \searrow 0$ pointwise on M imply $J(z_n) \to 0$. The functional J is extended by the usual Daniell definition to a larger function space L. (See also the more general treatment by the reviewer in the paper reviewed below.) A measure is recaptured from the Daniell integral J by defining $\mathfrak{A}$ as the family of all $A \subset M$ such that $\chi_A \in L$, and $\mathfrak{B}$ as the σ-algebra generated by A. The set-function $J(\chi_A)$ is a countably additive measure μ on $\mathfrak{A}$; it is extended as usual to $\mathfrak{B}$. Curiously enough, no mention is made of reobtaining J as the Lebesgue integral with respect to μ. (This has been done in one case by the reviewer and H. S. Zuckerman [Nagoya Math. J. **3**, 7–22 (1951); these Rev. **14**, 362].)

E. Hewitt (Seattle, Wash.).

Hewitt, Edwin. Integral representation of certain linear functionals. Ark. Mat. **2**, 269–282 (1952).

The purpose of this paper is to show that an integral representation of a linear functional I defined on a nonvoid linear space $\mathfrak{F}$ of real functions on a set X is possible under very modest assumptions on $\mathfrak{F}$ and I. If f and g denote arbitrary elements of $\mathfrak{F}$, the author only assumes that, for every nonnegative constant c, both min (f, g) and min (f, c) belong to $\mathfrak{F}$, and that I is nonnegative on nonnegative elements of $\mathfrak{F}$. Then he proceeds to construct a finitely additive measure γ^* defined on a ring $\mathfrak{M}$ of sets so that every f is $\mathfrak{M}$-measurable and so that the integral representation $I(f) = \int_X f d\gamma^*$ holds for every bounded f which vanishes outside some set of finite γ^*-measure. However, the proof of the auxiliary theorem 2.17 is deficient. According to a communication received from the author, the deficiency can be removed by extending the functional I, in accordance with a procedure of F. Riesz [Ann. of Math. **41**, 174–206 (1940), esp. pp. 203–205; these Rev. **1**, 147], so as to make integrable all characteristic functions of sets of finite γ-value in a class $\mathcal{P}$ of sets defined in the paper.

The paper ends with several applications and with a discussion of relations with results of other authors.

H. M. Schaerf (St. Louis, Mo.).

Yablonskiĭ, S. V. On convergent sequences of continuous functions. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk **1950**, no. 9, 13–29 (1950). (Russian)

Let $\{f_n\}$ be a sequence of real-valued, continuous functions defined on an interval I; let F_n and F be the graphs of f_n and f; let $\bar{F}$ [$\underline{F}$], the outer [inner] limit set of $\{F_n\}$, be the set of points of the plane each neighborhood of which meets an infinite number [all but a finite number] of the F_n. If $y_0 \geqq f(x_0)$ [$y_0 \leqq f(x_0)$] say that y_0 is the upper [lower] fringe limit of $\{f_n\}$ at x_0 if (A) given $\epsilon > 0$ for all but a finite number of n there exist x_n such that $(x_n, f_n(x_n))$ is within ϵ of (x_0, y_0); (B) for each $\eta > 0$ there exist $\delta > 0$ and N such that $f_n(x) < y_0 + \eta$ [$f_n(x) > y_0 + \eta$] if $n > N$ and $|x - x_0| < \delta$. If (B) [(A)] holds for $\{f_n\}$ and (A) [(B)] holds for some subsequence, y_0 is the weak [strong] fringe limit of $\{f_n\}$ at x_0 (upper or lower, as the case may be).

Theorem 1. If M is a set of the form

$$\{(x, y) \mid \varphi(x) \geqq y \geqq \psi(x),\ x \in I\},$$

then $M = \bar{F}$ [$M = \underline{F}$] if and only if φ and ψ are upper and lower semi-continuous, respectively, and, for each $x_0 \in I$, $\varphi(x_0)$ and $\psi(x_0)$ are the weak [strong] upper and lower fringe limits of $\{f_n\}$ at x_0. Theorem 2. In order that for three functions $\varphi \geqq f \geqq \psi$ there exist a sequence $\{f_n\}$ of continuous functions converging pointwise to f with upper and lower fringe limits φ and ψ, it is necessary and sufficient that (1) f be a function of the first Baire class, (2) φ and ψ be upper and lower semi-continuous, respectively, and (3) $\{x \mid \varphi(x) \neq \psi(x)\}$ be of first category.

If the conditions of Theorem 2 are satisfied by two pairs of functions $\bar{\varphi} \geqq \underline{\varphi} \geqq f \geqq \underline{\psi} \geqq \bar{\psi}$, then $\{f_n\}$ and $\{g_n\}$ with fringe limits $\bar{\varphi}$, $\bar{\psi}$ and $\underline{\varphi}$, $\underline{\psi}$ can be found; then the interlaced sequence, $f_1, g_1, \cdots, f_n, g_n, \cdots$, has weak [strong] fringe limits $\bar{\varphi}$ and $\bar{\psi}$ [$\underline{\varphi}$ and $\underline{\psi}$]. *M. M. Day* (Urbana, Ill.).

du Plessis, Nicolaas. A theorem about fractional integrals. Proc. Amer. Math. Soc. **3**, 892–898 (1952).

The theorem states that if $f \in L[0, 2\pi]$, then (a) for $0<\alpha<1$, $2<q<\infty$, $f_{\alpha/q}$ is finite everywhere except in a set which is of zero β-capacity for every $\beta>1-\alpha$, (b) for $0<\alpha<1$, $1 \leqq q \leqq 2$, $f_{\alpha/q}$ is finite everywhere except in a set of zero $(1-\alpha)$-capacity. Both statements are best possible. The fractional integrals are to be taken in either the Riemann-Liouville or the Weyl sense. *L. S. Bosanquet.*

Mil'man, D. On integral representations of functions of several variables. Doklady Akad. Nauk SSSR (N.S.) **87**, 9–10 (1952). (Russian)

Let Q be a bounded, closed subset of Euclidean n-space and let R be a linear space of real, continuous functions on Q such that R contains the constants and enough functions to separate points of Q. Using ideas from some of his earlier papers [same Doklady (N.S.) **57**, 119–122 (1947); **59**, 1045–1048 (1948); **83**, 357–360 (1952); these Rev. **9**, 192, 449; **13**, 848], he asserts that there exists a minimal closed subset Γ, the T-frontier of Q, uniquely determined by R and Q, such that to each q_0 in Q there is a normalized measure $\sigma(I, q_0)$ on Γ such that $x(q_0) = \int_\Gamma x(q) d\sigma(I_q, q_0)$ if $x \in R$. Many such measures may correspond to a single q_0; it is asserted that on certain subsets, called normal parts of Γ, all such measures coincide. An application of the theory is to harmonic functions on a bounded, closed region. *M. M. Day.*

Theory of Functions of Complex Variables

Heffter, Lothar. Gleichmässige Differenzierbarkeit einer Funktion und Stetigkeit ihrer Ableitung in einem Bereich. Arch. Math. **3**, 257–261 (1952).

A proof of the known theorem [cf., e.g., A. Pringsheim, Vorlesungen über Zahlen- und Funktionenlehre, Bd. II, Abt. 1, Teubner, Leipzig and Berlin, 1925, §53]: If the function $f(z)$ of a complex variable z is uniformly differentiable in the connected region G, then $f'(z)$ is continuous in G, and conversely. The author arrives at this theorem as the result of a series of lemmas concerning uniform differentiability of real functions of real variables.
T. A. Botts (Charlottesville, Va.).

Adel'son-Vel'skiĭ, G. M., and Kronrod, A. S. On a direct proof of the analyticity of a monogenic function. Doklady Akad. Nauk SSSR (N.S.) **50**, 7–9 (1945). (Russian)

It is shown that if a function $f(z) = u(x, y) + v(x, y)$ is monogenic in a region G bounded by a Jordan curve C (i.e., if $f(z)$ possesses a derivative at each point of G), then $f(z)$ is analytic in G in the sense that $f(z)$ can be expanded into a Taylor's series about each point of G. In the proof, the object of which is to avoid Cauchy's theorem, it is shown first that $|\operatorname{grad} u|_{(x_0, y_0)} \leqq 11\, Kr^{-1}$ and $|\operatorname{grad} v|_{(x_0, y_0)} \leqq 11\, Kr^{-1}$, where r is the distance of (x_0, y_0) from C and K is the maximum of the oscillations of u and v in G. It is then shown that the derivative of $f(z)$ is continuous in G and satisfies a Lipschitz condition with constant $392 \cdot 2^{1/2} Kr^{-2}$ in any region G' lying inside G whose distance from C is r. From this and from Arzelà's theorem it follows that the derivative of $f(z)$ is itself monogenic in G, and the rest is immediate.
A. J. Lohwater (Ann Arbor, Mich.).

Petracca, Antonio, and Levi, B. Study of a polydromic function. Math. Notae **11**, 124–138 (1951). (Spanish)

The function is $\sum_1^\infty z^{\lambda_n}$ where $\lambda_n = 1 + \frac{1}{2} + \cdots + 1/n$. The series is shown to be absolutely convergent if $|z| < 1/e$ but not if $|z| \geqq 1/e$, and divergent if $|z| \geqq 1$. Conditional convergence, however, is possible in the range $1/e \leqq |z| < 1$ by choice of the sequence of values of the many-valued functions z^{λ_n}. *A. J. Macintyre* (Aberdeen).

Varga, Richard S. Semi-infinite and infinite strips free of zeros. Univ. e Politecnico Torino. Rend. Sem. Mat. **11**, 289–296 (1952).

Let $f(z) = \sum a_n z^n$ be an entire function $\not\equiv 0$. Results of F. Carlson [Ark. Mat. Astr. Fys. **35A**, no. 14 (1948); these Rev. **10**, 27] imply that no sector can be entirely free of zeros of the partial sums $s_n(z)$ of the power series unless $f(z)$ is of zero order. The author imposes the conditions $a_n \geqq 0$, $n!a_n \geqq A(n+2)!a_{n+2}$ $(A>0)$, and shows by completely elementary methods that in this case the $s_n(z)$ have no zeros in the semi-infinite strip $|\operatorname{Im} z| \leqq (6A)^{\frac{1}{2}}$, $\operatorname{Re} z \geqq 0$. This result is applied to e^z; similar results applying to $\cos z$ and $\sin z$ are also proved. *J. Korevaar* (Madison, Wis.).

Valiron, Georges. Fonctions analytiques et équations différentielles. J. Math. Pures Appl. (9) **31**, 293–303 (1952).

By using refinements of a method previously employed by the author [Ann. Sci. Ecole Norm. Sup. (3) **38**, 389–429 (1921)] it is shown that no function $F(z)$ regular in $|z|<1$ and of infinite order in the unit circle, i.e., satisfying

$$\limsup_{r \to 1} \log\log M(r, F)/(-\log(1-r)) = \infty,$$

can satisfy a differential equation $P(z, F, F') = 0$, where P is a polynomial. The same conclusion also holds "in general" for algebraic differential equations of higher order.
W. H. J. Fuchs (Ithaca, N. Y.).

Rosenbloom, P. C. The fix-points of entire functions. Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 186–192 (1952).

L'auteur complète des résultats antérieurs [C. R. Acad. Sci. Paris **227**, 382–383 (1948); ces Rev. **10**, 187] sur l'itération. $f(z)$ est une fonction entière, $f_2(z) = f(f(z))$. Il établit que, d'après le théorème de Picard, si $f_2(z)$ n'a pas de point fixe, $f(z)$ est une translation; puis, utilisant les méthodes et notations de Nevanlinna, il établit que $f(z)$ et $g(z)$ étant fonctions entières, si $f(z)$ et $f(g(z))$ n'ont qu'un nombre fini de points fixes, ou bien $f(z)$ est un polynome, ou bien $g(z)$ une constante, ou $g(z) \equiv z$. Il prouve ensuite que, si $f(z)$ est un polynome de degré $k \geqq 2$ et $g(z)$ une fonction entière transcendante,

$$\limsup_{r \to \infty} \frac{N(r, 1/[f(g(z)) - z])}{T(r, g)} \geqq 1.$$

On a ce corollaire: si $f(z)$ est entière non linéaire et si $f(z)$ et $f(g(z))$ n'ont qu'on nombre fini de points fixes, $g(z)$ est un polynome. L'auteur en déduit ce théorème de Fatou: si une itérée de $g(z)$ n'a qu'un nombre fini de points fixes, $g(z)$ est un polynome. En relation avec ses résultats, l'auteur propose diverses recherches. *G. Valiron* (Paris).

Clunie, J. The determination of an integral function of finite order by its Taylor series. J. London Math. Soc. 28, 58–66 (1953).

En se bornant au cas des fonctions entières d'ordre fini, l'auteur donne une démonstration nouvelle du théorème sur le comportement d'une fonction entière $f(z)$ dans le voisinage des points de module maximum du cercle $|z|=r$. Contrairement à ce que fit Macintyre [Quart. J. Math., Oxford Ser. 9, 81–88 (1938)] l'auteur revient à la considération du terme maximum de la série de Taylor $\sum a_n z^n$ définissant $f(z)$, introduite dans ces questions par Borel et utilisée par Wiman et Valiron dans leurs recherches [voir Valiron, Lectures on the general theory of integral functions, Toulouse, 1923, chap. IV], puis par d'autres. N désigne le rang du plus grand des nombres $|a_n|r^n$, r étant donné. Le résultat principal est donné sous cette forme. On a

$$f(ze^\tau)=e^{N\tau}f(z)[1+\omega(\tau)],$$

avec $|\omega(\tau)|<K/\psi(N)$ pour

$$|\tau|<\frac{K}{N^{\frac{1}{2}}\log N\cdot\psi(N)},\quad K>\frac{1}{\psi(N)}>N^{-\gamma}\log^{-\frac{1}{2}}N,\quad \gamma>0,$$

si $|f(z)|>M(|z|)N^{-\beta}$, $\beta>0$, $|z|$ étant une valeur non exceptionnelle et $M(r)$ le maximum de $|f(z)|$ pour $|z|=r$. Dans ses démonstrations, l'auteur utilise une inégalité de Landau et une méthode de Saxer [Math. Z. 17, 206–227 (1923)]. *G. Valiron* (Paris).

Makar, Ragy H. Sur les suites de puissances fractionnaires de bases de polynomes. Bull. Sci. Math. (2) 76, 171–179 (1952).

A basic set of polynomials $\{p_n(z)\}$ is called simple if $p_n(z)$ is of degree n, monic if the coefficient of z^n is 1, and algebraic if the coefficient matrix P satisfies an algebraic equation. The author defines $\{p_n(z)\}^{r/s}$ as the set whose matrix is $P^{r/s}$, defined in the natural way when $\{p_n(z)\}$ is simple and monic. His principal result is that, when $\{p_n(z)\}$ is simple, monic and algebraic, $\{p_n\}^{r/s}$ has the same region of effectiveness as $\{p_n\}$; and that if $\{p_n\}$ is of order ω, $\{p_n\}^{1/s}$ is of order at least $\omega/(2s-1)$ but may be of arbitrarily large order. *R. P. Boas, Jr.* (Evanston, Ill.).

Walsh, J. L. Polynomial expansions of functions defined by Cauchy's integral. J. Math. Pures Appl. (9) 31, 221–244 (1952).

J. Plemelj [Monatsh. Math. Phys. 19, 205–210 (1908)] has shown: given a closed curve C satisfying some condition, and $f(z)$ defined and continuous on C, then $f(z)=f_1(z)-f_2(z)$, where the components $f_j(Z)$ are analytic functions of Z inside ($j=1$) or outside ($j=2$) of C,

$$f_j(Z)=(2\pi i)^{-1}\int_C[f(z)/(z-Z)]dz$$

and $f_j(Z)\to f_j(z)$ ($Z\to z$) for almost all z on C.

The problem was subsequently treated by several writers, e.g., J. Privaloff, J. L. Walsh, A. Ghika, E. C. Titchmarsh; for references, see this paper; also a paper by the reviewer [Amer. J. Math. 68, 398–416 (1946): these Rev. 8, 152] where C is replaced by the real axis. The problem plays a part in the approximation by rational functions, in integral equations, Fourier series and transformations, etc.

The author studies the subject under various hypotheses: C is an analytic Jordan curve, and in some results the origin is supposed to lie inside C; on C, $f(z)$ belongs to L^2, or to the Zygmund class $\Lambda^*(k)$, or to some Lipschitz class $L(k,\alpha)$ ($k=0,1,\cdots;0<\alpha<1$). In §1 the general theory of the components is treated, in §§2, 3 the main problem: the expansions of $f_j(Z)$ in Faber and Szegö polynomials and associated functions. Thus an interesting fact is proved (see §3, corollary 2; cf. §5): if $f(z)\in L(0,\alpha)$, and is not required to be analytic, yet the expansion of $f_1(Z)$ in Szegö polynomials converges uniformly in and on C. In §4 a particular property, relating to mutually orthogonal components, of the circle is discussed, and in §5 the degree of convergence of the polynomial expansions. *H. Kober.*

Gagua, M. E. On the behavior of analytic functions and their derivatives in closed regions. Soobščeniya Akad. Nauk Gruzin. SSR. 10, 451–456 (1949). (Russian)

Let E be a bounded set in the complex plane. A function $f(z)$ is called Λ_p-continuous on E if for arbitrary $z_1, z_2\in E$ we have

$$\text{(1)}\qquad |f(z_1)-f(z_2)|\leqq A|\log|z_1-z_2|^{-1}|^{-p},\quad A,p>0.$$

A function is called Λ_∞-continuous on E if (1) holds for all $p>0$ and with $A=A_p$ independent of z_i. Let D designate a simply connected region whose boundary L is a simple Jordan curve. The author studies the relationship between Λ_p-continuity of f and of f' on L and $D+L$. Using conformal mapping and a theorem of Walsh and Sewell [W. E. Sewell, Degree of approximation by polynomials in the complex plane, Princeton, 1942; these Rev. 4, 78] the following result is proved: If f is analytic in D, continuous in $D+L$ and is Λ_p-continuous on L, then $f(z)$ satisfies the inequality

$$|f(z)-f(z_0)|\leqq A_1|\log|z-z_0|^{-1}|^{-p},\quad z\in D+L,\quad z_0\in L.$$

For the case D: $|z|<1$, he proves that if f is analytic in D, continuous on L, and Λ_p-continuous on L, then

$$|f'(z)|\leqq A_1|\log(1-r)^{-1}|^{-p}(1-r)^{-1},\qquad r=|z|<1,\quad A,p>0.$$

Other theorems deal with Λ_p- continuity along a segment and with Λ_∞- continuity. *P. Davis* (Washington, D. C.).

Mergelyan, S. N. Uniform approximations of functions of a complex variable. Uspehi Matem. Nauk (N.S.) 7, no. 2(48), 31–122 (1952). (Russian)

This is an extensive study of three general problems in the theory of the uniform approximation of functions of a complex variable: the problem of uniform approximation by means of polynomials and rational functions on closed bounded sets, the uniform approximation of analytic functions on unbounded sets by means of entire functions, and the Bernstein-Jackson program for best (Tschebyscheff) approximation on arbitrary continua in the complex plane. This last is the problem of relating the asymptotic behavior of the sequence $\rho_n(f;E)$ of best approximations to f on a set E with the function-theoretic properties of E itself.

The proof of the author's theorem on the possibility of uniform approximation by polynomials [Doklady Akad. Nauk SSSR (N.S.) 78, 405–408 (1951); these Rev. 13, 23] is the principal object of the first chapter. This theorem states that in order for a function f defined on a closed set E to be approximated uniformly there it is necessary and sufficient that the complement of E consist of a single region containing the point at infinity and that f be continuous on E and analytic at each interior point of E. This is the culmination of a long series of results on uniform approximation by polynomials and includes the theorems due to Weierstrass, Runge, Walsh, Lavrentieff, and Keldysch. It is derived here from a theorem on the possibility of uniform approximation by meromorphic functions and this more

general question is investigated to a certain extent. The author gives several sufficient conditions on closed bounded sets E to admit of approximation by rational functions.

In the second chapter, the author considers the approximation of functions on unbounded sets E for which the following holds: (B) there exists a function $r(t)>0$, $\lim_{t\to\infty} r(t)=\infty$ such that an arbitrary point z of the complement of E can be connected with $z=\infty$ by a Jordan arc lying exterior to E and to the circle $|\zeta|<r(|z|)$. This is a necessary condition given by Keldysch and Lavrentieff [C. R. (Doklady) Acad. Sci. URSS (N.S.) **23**, 746–748 (1939); these Rev. **2**, 82] in order that E be a Carlemann continuum. The following general result is established with sharper results for sets E lying in angles and in strips: Let E be an arbitrary continuum satisfying (B). Let $f(z)$ be continuous at every finite point of E and analytic in the interior of E. Given arbitrary $\epsilon>0$ and $\eta>0$, there exists an entire function $G(z)$ for which $|f(z)-G(z)|<\epsilon\exp\{-|z|^{1-\eta}\}$ for $z\in E$. The growth of the approximating functions $G(z)$ is studied in detail.

The last chapter, containing inequalities for $\rho_n(f;E)$, is related to and generalizes some of the material in Sewell, Degree of approximation by polynomials in the complex domain, Princeton, 1942; these Rev. **4**, 78. Considered here are arbitrary bounded continua E with a connected complement. Theorems of both direct and inverse character are proved. *P. Davis* (Washington, D. C.).

Tonyan, V. A. On asymptotic approximation of continuous functions on sets separating the plane. Doklady Akad. Nauk SSSR (N.S.) **83**, 187–190 (1952). (Russian)

The approximations discussed are of the form

$$(1)\qquad |f(z)-H(z)|<\epsilon(|z|)$$

with H meromorphic, generalizing the work of Keldych and Lavrentieff [same Doklady (N.S.) **23**, 746–748 (1939); these Rev. **2**, 82] where this type of approximation was considered with H entire and on sets not separating the plane. The following sufficient condition is obtained. Let the closed set E be such that a circle of radius δ and center at an arbitrary point z contains a continuum of diameter $r(\delta)$ lying exterior to E. Let $r(\delta)$ be independent of z and $\lim\inf_{\delta\to 0}\delta/r(\delta)\leq k<\infty$. Then, if $f(z)$ is continuous on E with the possible exception of the point at infinity and if $\epsilon(r)$ is a continuous function which goes to zero with arbitrary rapidity as $r\to\infty$, there exists a meromorphic function $H(z)$ for which (1) holds for $z\in E$. The proof hinges upon a sufficient condition for the approximation of analytic functions by rational functions recently given by S. N. Mergelyan [see the paper reviewed above]. *P. Davis.*

Tonyan, V. A. On approximation of continuous functions on sets separating the plane. Akad. Nauk Armyan. SSR. Doklady **12**, 33–36 (1950). (Russian. Armenian summary)

A set M of the complex plane is called an α-set if an arbitrary continuous function defined on M can be expanded in a uniformly convergent series of rational functions. It is shown that if M has plane measure 0, then it is an α-set. On the other hand, an example is given of a closed, nowhere dense set which is not an α-set. *P. Davis.*

Leont'ev, A. F. On the completeness of a system of analytic functions. Mat. Sbornik N.S. **31**(73), 381–414 (1952). (Russian)

Let $f(z)=\sum_{k=0}^{\infty}a_{n_k}z^{n_k}$ $(a_{n_k}\neq 0)$ be an entire function; $\{\lambda_n\}_{n=1}^{\infty}$ a set of complex numbers. The set (1) $\{f(\lambda_n z)\}$ is called complete in a domain D not containing the origin if every function $\varphi(z)$ regular in D can be approximated by (1), i.e., $\varphi(z)$ is the uniform limit of a sequence (2) $\{P_n(z)=\sum_{j=1}^{p_n}\alpha_{nj}f(\lambda_j z)\}$ in every closed $E\subset D$. The set (1) is called complete in a domain D containing $z=0$ if every function $\varphi(z)$ with $\varphi^{(m)}(0)=0$, m non-ϵ $\{n_k\}$, can be approximated by (1). By means of the formula (A) $g(z)=\sum_{k=0}^{\infty}b_{n_k}z^{n_k}=(2\pi i)^{-1}\int\gamma(t/z)f(t)dt/z$, where

$$(3)\qquad \gamma(t)=\sum_{k=0}^{\infty}(b_{n_k}/a_{n_k})t^{-n_k-1}$$

and the integration is along a circle $|t|=\text{const.}$, the author compares the convergence of the sequence (2) and the sequences $\{Q_n(z)=\sum_{j=1}^{p_n}\alpha_{nj}g(\lambda_j z)\}$. Let R be the radius of the largest circle about 0 in which (1) is complete; R' has the analogous meaning for (4) $\{g(\lambda_n z)\}$. Theorem 1: If $\lim\sup_{k\to\infty}|b_{n_k}/a_{n_k}|^{1/n_k}\leq 1$, then $R'\geq R$. This is proved by using (A) to estimate $b_{n_k}z^{n_k}-Q_n(z)$ in terms of $a_{n_k}z^{n_k}-P_n(z)$. Similar theorems are obtained for completeness in other domains. For example: Theorem 2: If all singularities of (3) lie in the interval $\langle 0,1\rangle$, then (4) is complete in every domain which is star-shaped with respect to the origin and in which (1) is complete. In Theorems 3 to 6 it is assumed that $n_k=k$ $(k=0,1,2,\cdots)$ and $\lim_{n\to\infty}n^{1/\rho}|a_n|^{1/n}=(\sigma e\rho)^{1/\rho}$. This insures that $f(z)=\sum_0^{\infty}a_nz^n$ is of order ρ, type σ. Theorem 3: If $g(z)$ is of order ρ and type σ, $R'\geq R$. Theorem 4: No z^m $(m\geq 0)$ can be approximated by (1) in $|z|<R_1$, $R_1>R$. Theorem 5: If the coefficients of $g(z)$ satisfy

$$(5)\qquad \liminf_{n\to\infty} n^{1/\rho}|b_n|^{1/n}=(\tau e\rho)^{1/n},\quad \tau\leq\sigma,$$

then no z^m $(m\geq 0)$ can be approximated by (4) in $|z|<R_1$, $R_1>(\sigma/\tau)^{1/\rho}R$. It is shown by the example

$$g(z)=\sum_{n=0}^{\infty}a_{2n}z^{2n}+\sum_{n=0}^{\infty}(\tau/\sigma)^{(2n+1)/\rho}a_{2n+1}z^{2n+1},$$

$$\lambda_{2n-1}=-\lambda_{2n}=(cn)^{1/\rho}$$

that there are sets (4) satisfying (5) such that $R'=R$ while every odd power of z can be approximated by (4) in $|z|<(\sigma/\tau)^{1/\rho}R$.

It has been shown in an unavailable publication [Leont'ev, Trudy Mat. Inst. Steklov. **39** (1951)] that under the conditions (5) and $\lim\sup n/|\lambda_n|^{\rho}<\infty$ there is an r such that the hypothesis $F(z)$ can be approximated by (4) in $|z|<r$ implies: 1) $F(z)=\lim_{n\to\infty}Q_n(z)$, where $\alpha_{nj}\to\alpha_j$ as $n\to\infty$; 2) if $\alpha_j=0$ $(j=1,2,\cdots)$, then $F(z)\equiv 0$; 3) $r>R'$. The example just mentioned disproves the conjecture that any number greater than R' will do for r. Theorem 6: Hypothesis: $g(z)=g_1(z)+g_2(z)$ of order ρ, type σ; density of non-vanishing coefficients of $g_1(z)=\alpha$ $(0\leq\alpha\leq 1/2\rho)$, $g_2(z)$ of order ρ, type $\tau\leq\sigma$; $\lambda_n^{\rho}=\mu_n$, $\lim n/\mu_n=c>0$, $L(z)=\prod_{n=1}^{\infty}(1-(z/\mu_n)^2)$, $\lim_{n\to\infty}\mu_n^{-1}\log|L'(\mu_n)|=0$. Conclusion:

$$R'\geq\min\left\{\left(\frac{\pi c\cos\pi\rho\alpha}{\tau}\right)^{1/\rho},\ \left(\frac{\pi c}{\sigma\sin\pi\rho\alpha}\right)^{1/\rho}\right\}.$$

If $f(z)=f_1(z)+f_2(z)$ where no power of z occurs simultaneously in the expansions of $f_1(z)$ and $f_2(z)$, then the $\{f_j(\lambda_n z)\}$ $(j=1$ or $2)$ is complete in $|z|<R$, by Theorem 1. An example shows that the converse is not true. Another example shows that under the condition (5) R' may attain the value $(\sigma/\tau)^{1/\rho}R$. No larger value is possible, by Theorem 5.

W. H. J. Fuchs (Ithaca, N. Y.).

Potyagallo, D. B. On the set of boundary values of meromorphic functions. Doklady Akad. Nauk SSSR (N.S.) 86, 661–663 (1952). (Russian)

After definitions too long to be reproduced here the central result is stated in the following terms. If a linear continuum K can be exhibited in the form of a sum of linked elementary continua $K=K_1+K_2+\cdots+K_n+\cdots$ and K_n satisfies for sufficiently large n certain other conditions, then there exists a function $f(z)$ meromorphic in $|z|<1$ whose set of boundary values coincides with K. Generalisations to include continua which are not linear are described. The results are naturally more complicated than when a single boundary point on $|z|=1$ is under consideration. [Cf. L. Weigand, Comment. Math. Helv. 22, 125–149 (1949); these Rev. 10, 439.]

A. J. Macintyre (Aberdeen).

Tsuji, Masatsugu. An extension of Bloch's theorem and its applications to normal family. Tôhoku Math. J. (2) 4, 203–205 (1952).

The author's generalisation of Bloch's theorem to meromorphic functions [C. R. Acad. Sci. Paris 219, 301–303 (1944); these Rev. 7, 379] and results of Dufresnoy [these Rev. 7, 56] and Ahlfors [Acta Math. 65, 157–194 (1935)] are used to give very general criteria for normality of families of meromorphic functions. *A. J. Macintyre.*

Yûjôbô, Zuiman. An application of Ahlfors's theory of covering surfaces. J. Math. Soc. Japan 4, 59–61 (1952).

A new proof is given of the following theorem of Ahlfors [Acta Soc. Sci. Fennicae. Nova Ser. 2, no. 2 (1933)]. Let $w=f(z)$ be meromorphic for $|z|<R$. Let D_i $(i=1,\cdots,q)$ be disjoint simply connected domains in the w-plane and let the Riemann surface of the inverse of $f(z)$ be ramified μ_i-fold over D_i [cf. R. Nevanlinna, Eindeutige analytische Funktionen, Springer, Berlin, 1936, p. 336]. There exists a constant k depending only on the domains D_i such that, if $R|f'(0)|\geqq k(1+|f(0)|^2)$, then $q-\mu_1^{-1}-\cdots-\mu_q^{-1}\leqq 2$.

W. Kaplan (Ann Arbor, Mich.).

Hažaliya, G. Ya. On some covering theorems for functions regular in doubly connected regions. Akad. Nauk Gruzin. SSR. Trudy Mat. Inst. Razmadze 18, 245–256 (1951). (Russian. Georgian summary)

Let Σ be the class of functions $f(z)$ which are regular and single-valued in D_R: $1<|z|<R$, and for which $|f(z)|>1$, and $(2\pi i)^{-1}\int f'(z)f^{-1}(z)dz\geqq 1$ on the curves $|z|=c$ in D_R. The author defines the star of a doubly connected region in a manner too complicated to be reproduced here. It is his contention that if $f(z)\in\Sigma$, and if D^* is the star of D, the image of D_R under $f(z)$, then the area of D^* is not less than $\pi(R^2-1)$, and the length of the outer bounding curve of D^* is not less than $2\pi R$. The author's definition of the star of a doubly connected region is not meaningful to the reviewer, and the proofs seem to have serious gaps.

A. W. Goodman (Lexington, Ky.).

Nakashima, Katsuya. Note on subordination. Mem. Fac. Sci. Eng. Waseda Univ. 16, 119–122 (1952).

The author discusses the concept of subordination for univalent, and locally univalent, functions.

A. W. Goodman (Lexington, Ky.).

Sasaki, Yasuharu. On some family of multivalent functions. Kôdai Math. Sem. Rep. 1952, 89–92 (1952).

Functions of the form $F(z)=z^p+\sum_{p+1}^{\infty}a_nz^n$ are studied such that the image of $|z|<1$ is either (i) p-valent and star shaped, (ii) p-valent and convex, or (iii) p-valent and $|z|\leqq r$ $(0<r<1)$ corresponds to a curve of positive and finite curvature. A number of inequalities and relations between these classes are obtained and the conditions under which $|F(z)|\leqq M$ for $|z|<R$ imply that $F(z)$ belongs to class (iii) are obtained. *A. J. Macintyre* (Aberdeen).

Položil, G. N. On the motion of the boundary points of mapped regions. Uspehi Matem. Nauk (N.S.) 7, no. 6(52), 203–205 (1952). (Russian)

Let G be a simply-connected region lying inside another simply-connected region G^* in such a way that the frontiers of G and G^* have in common a Jordan arc Γ. It is shown that if G is mapped conformally onto G^* (with no other normalization as in the case of Löwner's lemma [cf., e.g., Nevanlinna, Eindeutige analytische Funktionen, Springer, Berlin, 1936, p. 51]), then there can be no more than three fixed points in the interior of the arc Γ. In the event that there are three such fixed points, the points of Γ gravitate toward the outer two fixed points under the conformal map and are repelled from the middle fixed point; while in the case of only two fixed points, one of the points attracts and the other repels. A number of immediate corollaries are stated. *A. J. Lohwater* (Ann Arbor, Mich.).

Huron, R. Sur un lemme de representation conforme. Ann. Fac. Sci. Univ. Toulouse (4) 15, 155–160 (1951).

By the standard method of comparison of length and area through Schwarz's inequality, the author makes estimates of conformal mapping on the boundary which have applications in the theory of free boundary flows.

P. R. Garabedian (Stanford, Calif.).

af Hällström, Gunnar. On the conformal mapping of incision domains. Soc. Sci. Fenn. Comment. Phys.-Math. 16, no. 13, 13 pp. (1952).

The author considers the conformal mapping of the unit circle F onto a domain E consisting of the unit circle provided with infinitely many radial slits. The capacity and Green's function of the point set on the boundary of F not corresponding to isolated slits bounding E are discussed.

P. R. Garabedian (Stanford, Calif.).

Strebel, Kurt. Über das Kreisnormierungsproblem der konformen Abbildung. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 101, 22 pp. (1951).

Proofs of the uniqueness, and also of the existence, of the conformal mapping of certain domains of infinite connectivity onto canonical domains bounded by circles and points. *P. R. Garabedian* (Stanford, Calif.).

*Szegö, G. **Recent contributions of the Hungarian school to conformal mapping.** Construction and applications of conformal maps. Proceedings of a symposium, pp. 267–268. National Bureau of Standards, Appl. Math. Ser., No. 18, U. S. Government Printing Office, Washington, D. C., 1952. $2.25.

*Poritsky, H. **Some industrial applications of conformal mapping.** Construction and applications of conformal maps. Proceedings of a symposium, pp. 15–30. National Bureau of Standards, Appl. Math. Ser., No. 18, U. S. Government Printing Office, Washington, D. C., 1952. $2.25.

Fourès, L. Recouvrements de surfaces de Riemann. Ann. Sci. Ecole Norm. Sup. (3) 69, 183–201 (1952).

The author introduces the following definition of a covering surface of a Riemann surface. Let t be a continuous mapping of a surface S^* into another surface S, such that $t^{-1}(M)$ for $M \in S$ is a set of isolated points. A point $P \in S$ is said to be covered by $\Delta \subset S^*$ if there exists a neighborhood $V(P) \subset S$ of P with the following properties. (1) $E_{V,\Delta} = \Delta \cap t^{-1}V(P)$ is not empty. (2) For any connected component Γ_i of $E_{V,\Delta}$, the intersection $P_i^* = \Gamma_i \cap t^{-1}(P)$ is not empty. (3) $\Gamma_i^* = \Gamma_i - P_i^*$ is a relatively nonramified covering of finite order of $V^*(P) = V(P) - P$. Then (S^*, t) is defined as a covering surface of S if every point $P \in S$ is covered by S^*. If S is a Riemann surface, this definition leads to the concept of a covering surface of a Riemann surface. The author discusses connections with the well-known definition of Stoïlow [Leçons sur les principes topologiques de la théorie des fonctions analytiques, Gauthier-Villars, Paris, 1937] and with the related notions of regularly ramified coverings, Riemann coverings, Riemann projections, and Riemann surfaces of given analytic functions. Automorphisms and regularly ramified topological trees are then studied, a typical result being as follows. With every simply connected Riemann surface R represented by a regularly ramified topological tree without infinite polygons are associated a closed Riemann surface R_1 and a mapping ω of R onto R_1 with the property that (R, ω) is a relatively nonramified covering of R_1. Finally, relations with Shimizu's theorem [Jap. J. Math. 8, 175–236 (1931); 237–304 (1932)] are discussed. *L. Sario* (Cambridge, Mass.).

Kusunoki, Yukio. Über die hinreichenden Bedingungen dafür, dass eine Riemannsche Fläche nullberandet ist. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 27, 99–108 (1952).

The author has an essentially new approach to the type problem. Let $\{F_n\}$ be an exhaustion of an arbitrary open Riemann surface F. Denote by $A_n(P)$ the function which maps F_n onto the λ_n-fold unit circle, a fixed point P_0 going into the origin [Ahlfors, Comment. Math. Helv. 24, 100–134 (1950); these Rev. 12, 90; 13, 1138]. Here $\lambda_n = \mu_n + 2\nu_n$, and μ_n, ν_n are the number of contours and the genus of F_n. The author sets $r_n = \sup_{F_0} |A_n(P)|$ and establishes that the divergence of the expression $(1/\lambda_n) \log (1/r_n)$ is a sufficient condition for the surface to be of parabolic type. The proof is given by estimating the Dirichlet integral of the harmonic measure. Connections with some other existence problems are also discussed. *L. Sario* (Cambridge, Mass.).

***Thullen, Peter. Problems of the theory of analytic functions of several complex variables.** Symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Diciembre, 1951, pp. 107–119. Centro de Cooperación Científica de la Unesco para América Latina, Montevideo, Uruguay, 1952. (Spanish)

Temlyakov, A. A. Analytic continuation of functions of two variables. Izvestiya Akad. Nauk. SSSR. Ser. Mat. 16, 525–536 (1952). (Russian)

Let $F(x, y) = \sum a_{mn} x^m y^n$ be a function of two complex variables analytic in the hypercone $|x| + |y| < R$. For rational q $(0 \leqq q \leqq 1)$ we put

$$f_q(u, v) = \sum_{m = q(m+n)} \frac{m!n!}{(m+n)!} u^m v^n.$$

If $0 \leqq \alpha < \beta \leqq 1$, the functions $f_{(\alpha,\beta)}(u, v) = \sum_{\alpha < q < \beta} f_q(u, v)$ and $f_{[\alpha,\beta]}(u, v) = \sum_{\alpha \leqq q \leqq \beta} f_q(u, v)$ are analytic in the bicylinder $|u| < R$, $|v| < R$. The author proves that $F(x, y)$ is singular at every point of the hypersurface $|x| + |y| = R$ if and only if the functions $f_{[0,\epsilon)}(u, v)$, $f_{(q-\epsilon, q+\epsilon)}(u, v)$, and $f_{(1-\epsilon,1]}(u, v)$ are singular at every point of the surface $|u| = |v| = R$ for arbitrarily small positive values of ϵ. A function analytic in the hypercone $|x| + |y| < R$ and represented by a gap series $\sum_{0 \leqq k \leqq n_p} a_{kp} x^k y^{n_p - k}$, where $n_p/p \to \infty$, is singular at every point of the hypersurface $|x| + |y| = R$ if

$$\limsup_{p \to \infty} \left| \sum_{q-\epsilon < k/n_p < q+\epsilon} \frac{k!(n_p - k)!}{n_p!} a_{kp} e^{-i(n_p - k)t} \right|^{1/n_p} = R^{-1}$$

for every t, rational q $(0 \leqq q \leqq 1)$, and positive ϵ [cf. Temliakoff, Mat. Sbornik N. S. 19(61), 73–84 (1946); these Rev. 9, 85]. *H. Tornehave* (Lyngby).

Stoll, Wilhelm. Mehrfache Integrale auf komplexen Mannigfaltigkeiten. Math. Z. 57, 116–154 (1952).

When analyzing theorems like Stokes formula $\int_{\partial B} \omega = \int_B \partial \omega$, say, in order to define the integrals occurring, it is appropriate to assume, in the first stages at least, that B is a union of simplices $\{s\}$ each of which satisfies certain differentiability requirements not only in its "interior" but also on its "closure". However, in dealing with point sets B which are defined as zero manifolds of analytic functions, it is not readily possible to extend the differentiability needed to the closure of the simplices, and it appears to be necessary to "exhaust" a simplex by a succession of approximating simplices from its interior, and correspondingly to define the integrals as a certain kind of improper integrals as it were.

Now, as a preliminary to a prospective investigation of zeros of meromorphic functions in several variables the author establishes several instances of such integrals and formulas, but his results are rather specialized for his purpose-to-come, and it would not be convenient to describe them on their own account. *S. Bochner.*

Bencivenga, Ulderico. Di alcuni sistemi metrici nello spazio a tre dimensioni. Rend. Accad. Sci. Fis. Mat. Napoli (4) 15 (1948), 54–81 (1949).

This paper, together with the following paper, contains an extension of the author's earlier results for the plane [Atti. Accad. Sci. Napoli (3) 2, no. 7 (1946); these Rev. 9, 26] to analogous problems in 3-space. Let $f(x_1, x_2, x_3)$ represent one of the four reducible cubic polynomials $(3^{1/2}/2)x_1(x_2^2 + x_3^2)$, $3^{3/2}x_1x_2x_3$, $3^{3/2}x_1x_2^2$, $3^{3/2}x_1^3$, or their orthogonal transformations. If a segment OP cuts the surface $f(x_1, x_2, x_3) = \pm 1$ in a point Q then the modulus of OP or of any segment parallel to OP and of equal length is defined to be $|OP/OQ|$. The modulus of an arc is defined by an integral in the obvious way. The author develops a parametric representation $x_1 = x_1(\mu, \gamma)$, $x_2 = x_2(\mu, \gamma)$, $x_3 = x_3(\mu, \gamma)$ in terms of trigonometric and exponential functions for points of the surface $f(x_1, x_2, x_3) = \pm 1$ in which μ is the modulus of a parametric line. These representations are obtained by a general method which makes use of certain solutions of third order linear homogeneous differential equations with constant coefficients. The solutions involved behave in a way analogous to the solutions of second order linear homogeneous differential equations with respect to the polynomials $x_1^2 + x_2^2$ and $x_1^2 - x_2^2$. A large number of special properties of the surface $(3^{1/2}/2)x_1(x_2^2 + x_3^2) = \pm 1$ is given.

A modulus for a plane area S is defined as S/M where M is the area of the region bounded by $f(x_1, x_2, x_3)=\pm 1$ and within the plane through the origin which is parallel to the plane of S. This concept is applied to obtain a relationship which connects the moduli of the three faces of a tetrahedron.
D. Derry (Vancouver, B. C.).

Bencivenga, Ulderico. Sulla rappresentazione geometrica di alcune algebre reali del 3° ordine. Rend. Accad. Sci. Fis. Mat. Napoli (4) **15** (1948), 131–151 (1949).

This paper, a sequel to the one reviewed above, deals with analysis in 5 commutative hypercomplex systems of rank 3 over the real field. The topics discussed include differentiable functions, convergence of power series, a counterpart of the Cauchy integral theorem and a modified conformal mapping. *D. Derry* (Vancouver, B. C.).

Bencivenga, Ulderico. Sulla rappresentazione geometrica delle algebre doppie dotate di modulo. II. Bicomplessi. Rend. Accad. Sci. Fis. Mat. Napoli (4) **18** (1951), 245–258 (1952).

This paper contains an extension of the author's results for bireal numbers [Atti Accad. Sci. Napoli (3) **2**, no. 7 (1946); these Rev. **9**, 26] to bicomplex numbers $z=z_1v_1+z_2v_2$ where z_1, z_2 are complex and v_1, v_2 are the matrices $\binom{1\ 0}{0\ 0}$, $\binom{0\ 0}{0\ 1}$ respectively. If $z_1=r_1e^{i\omega_1}$, $z_2=r_2e^{i\omega_2}$, k is defined to be the greatest multiple of π which does not exceed $\omega_1+\omega_2$. For each bicomplex number z two complex numbers R, Ω, known as its modulus and argument respectively, are defined for which $z=\pm R(e^{i\Omega}v_1\pm e^{-i\Omega}v_2)$ where the signs are determined by functions of k modulo 4.

The numbers z are represented geometrically on a system of two Gauss planes in which the second plane is superposed on the first so that its positive real axis coincides with the positive imaginary axis of the first. The bicomplex number $z=z_1v_1+z_2v_2$ is associated with the parallelogram defined by the origin, the points represented by z_1, z_2 in the first and second planes respectively and the point represented by z_1+z_2. The loci of z_1+z_2 are determined in the cases where one of R, Ω and one of $|z_1|$, $\arg z_1$ are simultaneously held constant. The functions $f_1(z_1)v_1+f_2(z_2)v_2$, where $f_1(z_1)$, $f_2(z_2)$ are analytic functions of z_1 and z_2 respectively, are discussed briefly with reference to the above geometric representation. *D. Derry* (Vancouver, B. C.).

Theory of Series

Agnew, Ralph Palmer. Inclusion relations among methods of summability compounded from given matrix methods. Ark. Mat. **2**, 361–374 (1952).

For each $r=1, 2, \cdots$ let $A(r)=(a_{nk}(r))$ denote a triangular and regular matrix of non-negative elements, which has an inverse. Let $B(r)$ denote the product matrix $A(r)A(r-1)\cdots A(2)A(1)$. For a given sequence $p=\{r_n\}$ of positive integers let $B[p]$ be the "compounded" matrix whose nth row coincides with the nth row of $B(r_n)$. The author proves the following theorems. (i) There is a sequence $1\leqq R_1\leqq R_2\leqq\cdots\to\infty$ such that $B[p]$ is regular if $1\leqq r_n\leqq R_n$ for each n. (ii) If p is nondecreasing, if $p'=\{r_n'\}$ is such that $r_n'\geqq r_n$ for all n sufficiently large, and if $B[p]$ and $B[p']$ are regular, then $B[p']$ includes $B[p]$. (iii) There is a sequence $1\leqq R_1\leqq R_2\leqq\cdots\to\infty$ such that $B[p]$ includes each of $B(1), B(2), \cdots$ if $1\leqq r_n\leqq R_n$ and $r_n\to\infty$. (iv) The regular transformations $B[p]$ corresponding to nondecreasing p form a consistent family. Applications are given which involve the methods of Cesàro, Abel, Euler, Borel, and Nörlund. Typical of these is the following result. (v) If $\sum u_n$ is summable to s by the classical Abel method, then there exists a regular compounded Cesàro method by means of which $\sum u_n$ is also summable to s. Finally, the author remarks that his work clarifies and extends results of Rudberg [Ark. Mat. Astr. Fys. **30A**, no. 10 (1944); these Rev. **7**, 152] based on similar ideas. *J. D. Hill.*

Martin, C. F. A brief proof of a theorem on T-transformations. Amer. Math. Monthly **60**, 29–30 (1953).

Es handelt sich um folgenden Satz: $\{x_k\}$ sei eine reelle Folge, $L=\liminf x_k$ und $U=\limsup x_k$, und $x\in\langle L, U\rangle$. Dann gibt es eine reguläre Matrix (a_{nk}) mit $a_{nk}\geqq 0$, so dass $\lim_{n\to\infty}\sum_{k=0}^{\infty}a_{nk}x_k=x$ ist. Frühere Beweise [vgl., z. B., R. G. Cooke, Infinite matrices and sequence spaces, Macmillan, London, 1950, S. 77–79; diese Rev. **12**, 694] enthielten Fallunterscheidungen, je nachdem L, U endlich oder $\pm\infty$ sind; die vom Verf. gegebene Darstellung vermeidet dies.
D. Gaier (Stuttgart).

Peyerimhoff, Alexander. Über einen Satz von Herrn Kogbetliantz aus der Theorie der absoluten Cesàroschen Summierbarkeit. Arch. Math. **3**, 262–265 (1952).

A short proof of the following extension of theorems of Kogbetliantz [Bull. Sci. Math. (2) **49**, 234–256 (1925)] is given. Let α and β be real numbers, not necessarily integers, for which $0\leqq\beta\leqq\alpha$. If $\sum a_n$ is evaluable $|C_\alpha|$ and $\gamma\geqq\alpha-\beta$, then the series $\sum a_n/(n+1)^\gamma$ and $\sum a_n/A_n{}^\gamma$ are evaluable $|C_\beta|$. It is remarked that the same method establishes analogous results for absolute Euler-Knopp evaluability.
R. P. Agnew (Ithaca, N. Y.).

Sunouchi, Gen-ichiro, and Tsuchikura, Tamotsu. Absolute regularity for convergent integrals. Tôhoku Math. J. (2) **4**, 153–156 (1952).

The authors prove that a transformation

$$\alpha(x)=\int_0^\infty b(x,t)a(t)dt,\quad \alpha=U(a),$$

is defined almost everywhere for each $a(t)\in L(0,\infty)$ and gives an $\alpha(x)\in L(0,\infty)$ if and only if

$$\operatorname*{ess.\,sup}_{0<t<\infty}\int_0^\infty|b(x,t)|dx\leqq M.$$

The only difficulty is to prove that $\alpha=U(a)$ is a bounded linear operation, since the general form of such operations from L to itself and their norms are known, and this is achieved by means of a lemma by Gelfand [Zapiski Naučno-Issled. Inst. Mat. Meh. i Har'kov. Mat. Obšč. (4) **13**, 35–40 (1936)]. *G. G. Lorentz* (Toronto, Ont.).

Evans, Arwel. The application of complex variable methods to Tauberian theorems. J. London Math. Soc. **28**, 94–102 (1953).

Der funktionentheoretische Teil der Arbeit wird durch folgenden Satz beleuchtet: Sei $h(z)$ in $\Re(z)>0$, regulär und beschränkt, ferner $h(z_n)\to\lambda$ für eine Punktfolge (z_n) mit $|z_n|\to 0$, $|z_n/z_{n+1}|\to 1$, $\limsup|\arg z_n|<\pi/2$. Dann ist $\lim h(z)=\lambda$, wenn z im Winkelraum $|\arg z|\leqq\alpha_1<\pi/2$ gegen 0 strebt. Beweis mit Hilfe des Auswahlprinzips (Satz von Montel). Anwendung dieses und eines verwandten Satzes

zum Beweis folgender Umkehrsätze für das Abelsche und Borelsche Verfahren. 1. Sei (z_n) eine Punktfolge der obigen Art, $F(z_n)=\int_0^\infty z_n e^{-z_n t}s(t)dt\to s$ $(n\to\infty)$, und $s(t)$ beschränkt [bzw. langsam oszillierend] in $\langle 0, \infty)$. Dann ist

$$T^{-1}\int_0^T s(t)dt\to s \quad (T\to\infty)$$

[bzw. $s(t)\to s$ $(t\to\infty)$]. Das Resultat in [ ] stammt von Delange [Ann. of Math. (2) **50**, 94–109 (1949); diese Rev. **10**, 368]. 2. Sei (z_n) eine Punktfolge mit $a\leqq\Im(z_n)\leqq b$ $(a<b)$, $\Re(z_n)>0$, $|z_n|\to\infty$, $|\Re(z_{n+1})-\Re(z_n)|\to 0$, und es sei $S(z_n)=e^{-z_n}\sum_{m=0}^\infty (z_n)^m A_m/m!\to s$ $(n\to\infty)$ sowie $a_n=O(n^{-1/2})$ mit $a_n=A_n-A_{n-1}$. Dann ist $A_n\to s$ $(n\to\infty)$. Beweis durch Zurückführung auf bekannte Umkehrsätze für das Abelsche und Borelsche Verfahren.

D. Gaier (Stuttgart).

Evgrafov, M. A. On an inverse of Abel's theorem for series having gaps. Izvestiya Akad. Nauk. SSSR. Ser. Mat. **16**, 521–524 (1952). (Russian)

Theorem. Hypothesis: 1) $f(z)=\sum_0^\infty a_n z^n$ is regular in $|z|<1$; 2) there is a sequence $\{n_k\}$ and a $\lambda>0$ such that $n_k\to\infty$, $a_n=0$ $(n_k<n\leqq n_k(1+\lambda))$; 3) as $z\to 1$ inside the circle $|z-x_0|<1-x_0$ $(0\leqq x_0<1)$, $f(z)\to s$. Conclusion: $\sum a_n=s$.

An example is given to show that the conclusion does not hold if hypothesis 3 is replaced by $f(z)\to s$ as $z\to 1$ inside $|\arg(1-z)|\leqq\alpha<\pi/2$.

W. H. J. Fuchs (Ithaca, N. Y.).

Evgrafov, M. A. Behavior of the power series for functions of class H_δ on the boundary of the circle of convergence. Izvestiya Akad. Nauk SSSR. Ser. Mat. **16**, 481–492 (1952). (Russian)

The paper deals with functions $f(z)=\sum_0^\infty a_n z^n$ regular and of class H_δ with $0<\delta<1$ in $|z|<1$ [cf. Zygmund, Trigonometrical series, Warsaw-Lvov, 1935, p. 158]. The main results are as follows. If $\delta=1/p$ with p a positive integer, then the power series is Cesàro summable order $p-1+\epsilon$ to $f(e^{i\theta})$ for almost all θ. If $\delta\geqq 1/p$, then the Cesàro "partial sums" of order $p-1+\epsilon$ are bounded on $z=e^{i\theta}$ by a function $F(\theta)$ of L_δ and converge in mean to $f(e^{i\theta})$. The coefficients are $o(n^{1/\delta-1})$ (this result is ascribed to G. A. Fridman) but not necessarily $O\{\psi(n)n^{1/\delta-1}\}$ for any prescribed $\psi(n)$ tending monotonically to zero. For series with Hadamard gaps there is a certain relation between the Cesàro sums and a subsequence of ordinary partial sums so that in the above theorems this subsequence can replace the sequence of Cesàro sums. For coefficients "near the gaps" the above order relation is strengthened.

A. J. Macintyre.

Norris, M. J. Some necessary conditions for convergence of infinite series and improper integrals. Amer. Math. Monthly **60**, 96–97 (1953).

Bohr, Harald. A study on the uniform convergence of Dirichlet series and its connection with a problem concerning ordinary polynomials. Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 21–34 (1952).

Same as Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **21**, no. 12 (1952): these Rev. **13**, 636.

Fourier Series and Generalizations, Integral Transforms

Tureckiĭ, A. H. On an extremal property of trigonometric polynomials satisfying a differential relation at isolated points of the interval. Doklady Akad. Nauk SSSR (N.S.) **50**, 65–68 (1945). (Russian)

Let $P_n(x)$ be a trigonometric polynomial of degree n. The author determines, by means of an interpolation formula, the maximum of $|P_n(x)|$ given that

$$|aP_n''(x_k)+bP_n'(x_k)+cP_n(x_k)|\leqq 1,$$

where $x_k=2k\pi/(2n+1)$. He discusses the asymptotic behavior of the maximum for various relationships among a, b, c.

R. P. Boas, Jr. (Evanston, Ill.).

Sunouchi, Gen-ichiro. Convergence criteria for Fourier series. Tôhoku Math. J. (2) **4**, 187–193 (1952).

Let $\varphi(x)$ be even and periodic of period 2π. Let $\eta>0$ be fixed and let $d\geqq 1$. Then $\int_0^t\varphi(u)du=o(t^d)$ and

$$\lim_{k\to\infty}\lim_{x\to 0}\sup\int_{(kx)^{1/d}}^\eta\left|\frac{\varphi(t)}{t}-\frac{\varphi(t+x)}{(t+x)}\right|dt=0$$

imply the convergence of the Fourier series of $\varphi(x)$ at $x=0$. The same conclusion holds if $\int_0^t|\varphi(u)|du=o(t/\log(1/t))$ and either

$$\lim_{k\to\infty}\lim_{x\to 0}\sup\int_{(kx)^{1/d}}^\eta\left|\frac{\varphi(t)}{t}-\frac{\varphi(t+x)}{(t+x)}\right|dt=0$$

or

$$\lim_{n\to\infty}\lim_{x\to 0}\sup\int_{(kx)^{1/d}}^\eta\frac{|\varphi(x+t)-\varphi(t)|}{-t}dt=0.$$

The first result extends convergence criteria of S. Pollard [J. London Math. Soc. **2**, 255–262 (1927)] and G. Sunouchi [Tôhoku Math. J. (2) **3**, 216–219 (1951); these Rev. **13**, 739]. The second result extends that of Hardy and Littlewood [Ann. Scuola Norm. Super. Pisa (2) **3**, 43–62 (1934)].

P. Civin (Eugene, Ore.).

Misra, M. L. On the Cesàro summability of trigonometric series. Proc. Nat. Acad. Sci. India. Sect. A. **15**, 106–124 (1946).

Let $f(x)$ be a function integrable on $[-\pi, \pi]$ and of period 2π. Let $\psi_0(t)=f(x+t)-f(x-t)$, and $\psi_n(u)=u^{-1}\int_0^u\varphi_{n-1}(t)dt$. Several results concerning summability (C, k) of the conjugate and Fourier series are established, among which are the following. The conjugate Fourier series of $f(t)$ is summable (C, k), provided that the conjugate integral is summable $(C, [k])$ and (*) $t^{-1}\int_0^t|\psi_{[k]}(u)|du=O(1)$ when $k-[k]>0$. For points at which (*) and $t^{-1}\int_0^t\psi_{[k]}(u)du=o(1)$ hold, the summability $(C, [k])$ of the conjugate integral to S is necessary for the summability (C, k) of the conjugate series to S.

P. Civin (Eugene, Ore.).

Yano, Shigeki. Cesàro summability of Fourier series. J. Math. Tokyo **1**, 32–34 (1951).

Let $f(x)\in L$ have period 2π and let

$$\tfrac{1}{2}a_0+\sum_1^\infty(a_n\cos nx+b_n\sin nx)$$

be its Fourier series. The purpose of the paper is to prove the following theorem: If $f\in L^2$, the series

$$\sum_1^\infty n^{-\alpha/2}(a_n\cos nx+b_n\sin nx)$$

is summable $(C, -\tfrac{1}{2}\alpha)$ except in a set of $(1-\alpha)$-capacity

zero; if $f \in L^p$ $(p>1)$, the series $\sum_1^\infty n^{-\alpha/p}(a_n \cos nx + b_n \sin nx)$ is summable $(C, -\alpha/p)$ except in a set of $(1-\alpha+\epsilon)$-capacity zero for any ϵ such that $0<\epsilon<\alpha$. *R. Salem* (Paris).

Pati, T. On the absolute summability of the conjugate series of a Fourier series. Proc. Amer. Math. Soc. 3, 852–857 (1952).

Let $f(t)$ be an integrable function with period 2π and

$$\psi(t)=\tfrac{1}{2}[f(x+t)-f(x-t)],\quad \theta(t)=\frac{2}{\pi}\int_t^\infty u^{-1}\psi(u)du.$$

For $\alpha>0$, we define

$$\psi_\alpha(t)=\frac{\alpha}{t}\int_0^t\left(1-\frac{u}{t}\right)^{\alpha-1}\psi(u)du,$$

$$\theta_\alpha(t)=\frac{\alpha}{t}\int_0^t\left(1-\frac{u}{t}\right)^{\alpha-1}du\int_u^\infty\frac{\psi(v)}{v}dv$$

and $\psi_0(t)=\psi(t)$, $\theta_0(t)=\theta(t)$. Bosanquet and Hyslop [Math. Z. 42, 489–512 (1937)] have proved that, if $\alpha\geqq 0$, $\psi_\alpha(t)$ is of bounded variation in $(0,\pi)$ and $\theta_\lambda(t)$ is of bounded variation in $(0,\pi)$ for some λ, then the conjugate Fourier series of $f(t)$ is summable $|C,\alpha+\delta|$ for $\delta>0$. They have also proved that, in the case $\alpha=0$, the theorem is best possible, i.e., it does not necessarily hold for $\delta=0$. The author proves that the theorem is also best possible in the case $\alpha=1$ by the example:

$$f(t)\sim\sum_{n=2}^\infty\frac{\cos nt}{\log n}.$$

By this example, he also shows that convergence, together with $|A|$-summability, of the conjugate Fourier series do not imply $|C,1|$-summability in general. *S. Izumi.*

Hartman, Philip, and Wintner, Aurel. On sine series with monotone coefficients. J. London Math. Soc. 28, 102–104 (1953).

It is proved that if $\{b_n\}$ is a nonincreasing sequence of positive numbers tending to 0, and $f(x)=\sum_1^\infty b_n \sin nx$, then $f(x)/x$ tends to $\sum_1^\infty nb_n$ as $x\to 0$, if this series converges, and to infinity if the series diverges. *J. L. B. Cooper.*

Avadhani, T. V. On summation over lattice points. J. Indian Math. Soc. (N.S.) 16, 103–125 (1952).

This paper studies summability by Riesz' typical means of the series (1) $\sum_{n=0}^\infty n^\sigma r_k(n,h)$ and

$$(2)\qquad \sum_{n=0}^\infty n^\nu J_\nu(2\pi\xi\sqrt{n})r_k(n,h)\quad(\nu>-1),$$

where h is a vector $(h_1,\cdots,h_k)$, and

$$r_k(n,h)=\sum\exp(2\pi i(n_1h_1+\cdots+n_kh_k)),$$

the sum being taken over all lattice points $(n_1,\cdots,n_k)$ on the sphere $n_1^2+\cdots+n_k^2=1$. It is assumed that ξ is not equal to the distance of any point $(n+h)$ to the origin, and $\xi\neq 0$.

Using Poisson's summation formula and many lemmas on Bessel functions, the author obtains summability indices for (1) and (2). If h is not a lattice point, then (1) is summable $(R;n,\alpha)$ if $\sigma>0$, $\alpha>2\sigma+\frac{1}{2}(k-1)$, and if $-k(k-1)/2(k+1)\leqq\sigma<0$, $\alpha>\frac{1}{2}(k-1)+\sigma(1+1/k)$. If h is a lattice point, the series has positive terms only, and the author gives an asymptotic formula for the partial Riesz sums.

If h is a lattice point, the author's results concerning (2) are, roughly speaking, the same as those of Bochner and Chandrasekharan [Quart. J. Math., Oxford Ser. (1) 19, 238–248 (1948); (2) 1, 80 (1950); these Rev. 10, 431; 11, 646]. If h is not a lattice point, however, he improves the summability index given by these authors, roughly speaking, by an amount of $\frac{1}{2}(k-1)$ if $\sigma>0$. For $\sigma<0$ there is also an improvement, but it is more difficult to describe.

In all cases considered the author also gives an index for absolute summability by Riesz' means. These indices exceed those given for ordinary summability by an amount 1. *N. G. de Bruijn* (Amsterdam).

Szegö, G. On certain Hermitian forms associated with the Fourier series of a positive function. Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 228–238 (1952).

Let $f(\theta)\in L$ be real, have period 2π, and let $\{c_n\}$ be the sequence of its complex Fourier constants:

$$c_n=(2\pi)^{-1}\int_{-\pi}^\pi f(\theta)e^{-in\theta}d\theta\quad(\bar c_n=c_{-n}).$$

Let $D_n(f)$ be the determinant of the Hermitian form

$$H_n=\sum_{\mu,\nu=0,\cdots,n}c_{\nu-\mu}x_\mu\bar x_\nu,\quad(n=0,1,2,\cdots).$$

It has been conjectured by Pólya and proved by the author that if $f(\theta)\geqq 0$,

$$(1)\quad \lim_{n\to\infty}\{D_n(f)\}^{1/(n+1)}=\exp\left\{\frac{1}{2\pi}\int_{-\pi}^\pi\log f(\theta)d\theta\right\}=G(f).$$

The author now proves the following result: Suppose that $f(\theta)\geqq 0$ has a derivative $f'(\theta)$ which satisfies a Lipschitz condition of order α $(0<\alpha<1)$. Let

$$K(z)=\frac{1}{2\pi}\int_{-\pi}^\pi\log f(\theta)\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}d\theta=\sum_0^\infty K_nz^n$$

be the function regular for $|z|<1$, real at the origin, and whose real part has $\log f(\theta)$ as boundary value on the unit circle. One has $K_n=O(n^{-1-\alpha})$. Let $h(z)=\exp\{\frac{1}{2}K(z)\}$. Then, $G(f)$ being defined as in (1), one has

$$\lim\frac{D_n(f)}{[G(f)]^{n+1}}=\exp\left\{\frac{1}{\pi}\int\int\left|\frac{h'(z)}{h(z)}\right|^2d\sigma\right\},$$

the dou[illegible]ntegral being extended over the circle $|z|\leqq 1$ ($d\sigma$ elem[illegible]t of area). *R. Salem* (Paris).

Izumi, Shin-ichi. Notes on Fourier analysis. XLIV. On the law of the iterated logarithm of some sequences of functions. J. Math. Tokyo 1, 1–22 (1951).

Generalizing results of T. Tsuchikura, R. Fortet, and G. Maruyama, the author proves that if $f(x)$ is bounded, has period 1, mean value zero, and satisfies

$$\int_0^1\max_{0\leqq\nu\leqq u}|f(x+\nu)-f(x)|dx=O\left[1\Big/\left(\log\left(\frac{1}{u}\right)\right)^\alpha\right]$$

with $\alpha>1$, then for almost all t

$$(1)\qquad \limsup_{N\to\infty}\frac{\sum_1^N f(2^nt)}{(N\log\log N)^{1/2}}=\sigma^2,$$

where

$$\sigma^2=\lim_{N\to\infty}\frac{1}{N}\int_0^1\left(\sum_1^N f(2^nt)\right)^2dt.$$

The case $\alpha=1$ leads, under more complicated conditions, to a result in which (1) is replaced by an inequality. Applica-

tions are made to the study of the "discrepancy" in uniform distribution. *R. Salem* (Paris).

Ahiezer, N. I. On a proposition of A. N. Kolmogorov and a proposition of M. G. Krein. Doklady Akad. Nauk SSSR (N.S.) **50**, 35–39 (1945). (Russian)

Let $L^p(d\sigma; a, b)$ denote the linear space with norm given by $\|f\|^p=\int_a^b|f(t)|^p d\sigma(t)$, where $\sigma(t)$ is nondecreasing and bounded. The author proves that $\{e^{int}\}_0^\infty$ spans $L^p(d\sigma; -\pi, \pi)$, $p\geqq 1$, if and only if $\int_{-\pi}^{\pi}|\log \sigma'(t)|dt=\infty$; and that $\{e^{i\alpha x}\}$, $\alpha\geqq 0$, spans $L^p(d\sigma; -\infty, \infty)$ if and only if

$$\int_{-\infty}^{\infty}(1+x^2)^{-1}|\log \sigma'(x)|dx=\infty.$$

The content of this paper is reproduced in the author's book [Lectures on the theory of approximation, Moscow-Leningrad, 1947, pp. 278–283; these Rev. **10**, 33]. For further developments see Tumarkin [same Doklady (N.S.) **84**, 21–24 (1952); these Rev. **14**, 154]. *R. P. Boas, Jr.*

Tanaka, Chuji. Note on Laplace-transforms. VIII. On the singularity-criterion of Laplace-transforms. Jap. J. Math. **21** (1951), 29–35 (1952).

[For papers I–VII see these Rev. **13**, 840.] The author, using standard methods, gives a necessary and sufficient condition that a Laplace-Stieltjes transform $F(s)=\int_0^\infty e^{-st}d\alpha(t)$ with abscissa of convergence $\sigma_c=0$ be singular at $s=0$.
I. I. Hirschman, Jr. (St. Louis, Mo.).

Tanaka, Chuji. Note on Laplace-transforms. IX. On the singularities of Laplace-transforms. I. Jap. J. Math. **21** (1951), 37–42 (1952).

A number of corollaries are deduced from the result of the preceding paper. For example, if the Laplace-Stieltjes transform $F(s)$ of $\alpha(t)$ has abscissa of convergence $\sigma_c=0$, then $s=0$ is a singularity for $F(s)$ if $\alpha(t)$ is monotone.
I. I. Hirschman, Jr. (St. Louis, Mo.).

Tanaka, Chuji. Note on Laplace-transforms. X. On the singularities of Laplace-transforms. II. Jap. J. Math. **21** (1951), 43–51 (1952).

The author studies the relations between the changes of sign of the $f(t)$ and the singularities of its Laplace transform $F(s)=\int_0^\infty e^{-st}f(t)dt$. Similar results have been given by H. Delange [C. R. Acad. Sci. Paris **233**, 1413–1414 (1951); these Rev. **13**, 551]. *I. I. Hirschman, Jr.*

Tanaka, Chuji. Laplace-transforms. XI. The singularities of Laplace-transforms. III. Duke Math. J. **19**, 605–613 (1952).

If a function $F(s)$ is holomorphic in the half-plane $\Re(s)>\Re(s_0)$ and if $F(s)$ takes on every value with at most two exceptions in each semi-circle $|s-s_0|<\delta$ in this half-plane, the author calls $s=s_0$ a Picard point of the function. If

$$F(s)=\int_0^\infty \exp(-sx)d\alpha(x)$$

converges for $\Re(s)>0$, the author shows that $s=0$ is a Picard point of $F(s)$ if

(i) $$\limsup_{m\to\infty}|O_m|^{1/m}\geqq 1,$$

(ii) $$\operatorname{im\,sup}_{m\to\infty}(\log m)^{-1}\log^+\log^+|O_m|>\tfrac{1}{2}$$

with

$$O_m=(e/m)^m\int_{m(1-\omega)}^{m(1+\omega)}x^m e^{-x}d\alpha(x),\quad 0<\omega<1.$$

These conditions are satisfied, in particular, if there exists a sequence $\{x_n\}$, $0<x_n\uparrow\infty$, such that

(a) $\Re[\alpha(x)]>0$ in $(1-\omega)[x_n]\leqq x\leqq(1+\omega)[x_n]$,

(b) $$\limsup_{n\to\infty}x_n^{-1}\log R_n=0,$$

(c) $$\limsup_{n\to\infty}(\log x_n)^{-1}\log^+\log^+ R_n>\tfrac{1}{2}$$

with $R_n=\Re\{\alpha(x_n)-\alpha([x_n])\}$. If $|\arg d\alpha(x)|\leqq\theta<\pi/2$, one can suppress (a) and replace R_n by $|\alpha(x_n)-\alpha([x_n])|$. Applications to Dirichlet series (extensions of Hadamard's gap theorem and Fekete's theorem). *E. Hille* (Nancy).

Tanaka, Chuji. Note on Laplace-transforms. XII. On the summability-abscisses of Laplace-transforms. Kōdai Math. Sem. Rep. **1952**, 77–88 (1952).

The Laplace transform $\int_0^\infty e^{-sx}d\alpha(x)$ is said to be (R, k) summable at $s=s_0$ if

$$\lim_{\omega\to\infty}\omega^{-k}\int_0^\omega(\omega-x)^k e^{-s_0x}d\alpha(x)$$

exists. The author proves that there exists an (R, k) abscissa of summability, a fact which has long been known [see Doetsch, Handbuch der Laplace-Transformation, Bd. I, Birkhäuser, Basel, 1950, Chapter IX; these Rev. **13**, 230]. A similar result is demonstrated for "absolute" (R, k) summability. Unfortunately "absolute" (R, k) summability is equivalent to absolute convergence. A treatment of "uniform" (R, k) summability is also given.
I. I. Hirschman, Jr. (St. Louis, Mo.).

Rooney, P. G. A new representation and inversion theory for the Laplace transformation. Canadian J. Math. **4**, 436–444 (1952).

The author shows that the operator:

$$L_{k,t}[f(s)]=(ke^{2k}(\pi t)^{-1})\int_0^\infty x^{-\frac{1}{2}}\cos(2kx^{\frac{1}{2}})f(k(x+1)/t)dx$$

which was also introduced in the same connection by Erdélyi [Math. Mag. **24**, 1–6 (1950); these Rev. **12**, 256] yields a "real" inversion formula for the Laplace transform. That is, if $f(s)=\int_0^\infty\exp(-st)\phi(t)dt$, then under suitable conditions $\lim L_{k,t}[f(s)]=\phi(t)$ for $k\to\infty$. Representation theorems are also derived and the results are extended to the Laplace-Stieltjes transform. *S. Agmon* (Jerusalem).

San Juan, Ricardo. Caractérisation directe sous forme exponentielle des transformations de Laplace généralisées. Portugaliae Math. **11**, 105–118 (1952).

The author continues his study of the properties which characterize the Laplace transform [see, for example, Portugaliae Math. **9**, 177–184 (1950); these Rev. **12**, 406]. Let $\mathfrak{F}_z[\phi]$ be a functional on a certain linear space S of functions defined on $[0, \infty]$, for every z in some set D. It is desired to know when $\mathfrak{F}_z$ is of the form

$$\mathfrak{F}_z[\phi(t)]\int_0^\infty\phi(t)e^{-tg(z)}dt,$$

where $\operatorname{Re} g(z)\geqq 0$ for $z\in D$. Formally it is evidently necessary that (*) $\mathfrak{F}_z[\phi_1 * \phi_2]=\mathfrak{F}_z[\phi_1]\cdot\mathfrak{F}_z[\phi_2]$, and that (**) $g(z)\mathfrak{F}_z[\phi]=\mathfrak{F}_z[\phi']+\phi(0)$. Under suitable conditions (*) and (**) are each necessary and sufficient.
I. I. Hirschman Jr. (St. Louis, Mo.).

***Doetsch, Gustav.** **Unsolved problems in the theory of the Laplace transform.** Symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Diciembre, 1951, pp. 169–176. Centro de Cooperación Científica de la Unesco para América Latina, Montevideo, Uruguay, 1952. (Spanish)

***Laguardia, Rafael.** **Problems in the iteration of the Laplace transform.** Symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Diciembre, 1951, pp. 177–183. Centro de Cooperación Científica de la Unesco para América Latina, Montevideo, Uruguay, 1952. (Spanish)

Bose, S. K. **Generalised Laplace integral of two variables.** Ganita 3, 23–35 (1952).

Continuing his earlier work [see these Rev. 11, 28, 173, 174; 12, 95, 256; 13, 31, 458], the author now considers double Whittaker transforms. He obtains the formal rules, computes some transform pairs, and obtains two results for the case that the Whittaker functions reduce to Laguerre polynomials. *A. Erdélyi* (Pasadena, Calif.).

Bhatnagar, K. P. **On a new relation in the theory of generalized Laplace integral.** Ganita 3, 13–18 (1952).

The author takes the second iterate of the Laplace transform of a Whittaker transform and writes out several examples of the ensuing relation. *A. Erdélyi.*

Dinesh Chandra. **On Whittaker transform.** Ganita 3, 1–11 (1952).

The author obtains two results concerning combinations of Whittaker and Laplace transforms, and illustrates these by examples. He also gives some examples of results by S. K. Bose. *A. Erdélyi* (Pasadena, Calif.).

Mitra, S. C., and Bose, B. N. **On certain theorems in operational calculus.** Acta Math. 88, 227–240 (1952).

Let $\phi(p)$ and $\psi(p)$ denote the operational correspondents of $f(t)$ and $g(t)$ in the terminology of Heaviside's operational calculus. Thus $\phi(p)=pL\{f(t)\}$ where L denotes the Laplace transformation with parameter p. From the known formula $\int_0^\infty x^{-1}\phi(x)g(x)dx=\int_0^\infty x^{-1}\psi(x)f(x)dx$ the operational correspondents of the function $t^{n-1/2}\int_0^\infty e^{-x^2t}f(x)dx$ and of certain other somewhat similar functions are written in terms of integrals involving $\phi(p)$ and Bessel functions. From these formulas the authors obtain results on self-reciprocal Hankel and Fourier transforms, a somewhat typical one of which is the following. If $t^{-n-1/2}f(t)$ is self-reciprocal with respect to the Hankel transformation on the semi-infinite interval $t>0$ with kernel J_n, then $t^{n-3/2}\phi(t)$ is self-reciprocal with respect to that transformation. Conditions of validity of most of the results are indicated but not established.

R. V. Churchill (Ann Arbor, Mich.).

Harmonic Functions, Potential Theory

Miranda, Carlo. **Sulla sommabilità delle derivate di una funzione armonica hölderiana.** Rend. Accad. Sci. Fis. Mat. Napoli (4) 18 (1951), 86–88 (1952).

Let D denote a domain in E^n, and let $H(P)$ be harmonic in D and satisfy a uniform Hölder condition with exponent μ. Then, by direct calculation, the author proves that H has first partials which are in class L^p over D, where $p<(1-\mu)^{-1}$. *M. O. Reade* (Ann Arbor, Mich.).

Allen, A. C. **Note on a theorem of Gabriel.** J. London Math. Soc. 27, 367–369 (1952).

L'A. si propone di dare una nuova dimostrazione di un teorema di R. M. Gabriel [Proc. London Math. Soc. (2) 34, 305–313 (1932)]. A noi pare che tale dimostrazione cada in difetto nella formula (B). Infatti l'A. fa le ipotesi seguenti: $U(t)$ sia una funzione positiva integrabile nell'intervallo $(-\pi,\pi)$; $u(r,\theta)$ sia l'integrale di Poisson della funzione $U(t)$; E sia un insieme contenuto nell'intervallo $(-\pi,\pi)$; $g(x)$ sia la funzione caratteristica dell'insieme E; infine sia

$$f(x)=\begin{cases} U(x) & \text{per } -\pi\leqq x\leqq\pi \\ 0 & \text{altrove,}\end{cases}$$

$$h(x)=\begin{cases} \dfrac{1-r^2}{1-2r\cos x+r^2} & \text{per } -\pi\leqq x\leqq\pi \\ 0 & \text{altrove.}\end{cases}$$

Quindi l'A. afferma che

$$\text{(B)}\quad \int_E u(r,\theta)d\theta=\int_{-\infty}^{+\infty}u(r,\theta)g(\theta)d\theta =\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(x)h(x-\theta)g(\theta)dxd\theta;$$

data l'arbitrarietà dell'insieme E, la formula (B) equivale alla

$$\text{(B}'\text{)}\qquad u(r,\theta)=\int_{-\infty}^{+\infty}f(x)h(\theta-x)dx\quad (\text{per } -\pi\leqq\theta\leqq\pi),$$

la (B′) equivale alla

$$\text{(B}''\text{)}\quad \begin{cases} u(r,\theta)=\displaystyle\int_{\theta-\pi}^{\pi}\frac{(1-r^2)u(x)}{1-2r\cos x+r^2}dx & (\text{per } 0\leqq\theta\leqq\pi) \\ u(r,\theta)=\displaystyle\int_{-\pi}^{\pi+\theta}\frac{(1-r^2)u(x)}{1-2r\cos x+r^2}dx & (\text{per } -\pi\leqq\theta\leqq 0)\end{cases}$$

e la (B″) é evidentemente errata. *E. De Giorgi.*

Vekua, Ilia. **On a generalization of the Poisson integral for a half-plane.** Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 15, 149–154 (1947). (Georgian. Russian summary)

It is proved that the formula

$$u(x,y)=\frac{\Gamma\left(\dfrac{k+2}{2k+2}\right)(1+k)^{1/(1+k)}}{\Gamma\left(\dfrac{1}{2}\right)\Gamma\left(\dfrac{1}{2k+2}\right)}y \times\int_{-\infty}^{\infty}f(t)\left[(x-t)^2+\frac{y^{2k+2}}{(k+1)^2}\right]^{-(k+2)/(2k+2)}dt$$

gives the solution to the Dirichlet problem for the equation $y^{2k}\partial^2u/\partial x^2+\partial^2u/\partial y^2=0$ in the case of the half-plane $y>0$.

Author's summary.

Kasner, Edward, and De Cicco, John. **The Newtonian potential of a sphere of a Euclidean space E_N embedded in a Euclidean universe E_n.** Proc. Nat. Acad. Sci. U. S. A. 38, 905–911 (1952).

Calcul de la force d'attraction newtonienne dans l'espace à n dimensions d'un corps homogène en forme d'un domaine ou surface sphérique de dimension $N\leqq n$. Expressions au moyen des fonctions hypergéométriques. *M. Brelot.*

Jost, Res, and Kohn, Walter. Construction of a potential from a phase shift. Physical Rev. (2) **87**, 977–992 (1952).

The paper is mainly devoted to constructing the potential, in the form of a series, from a given S phase shift. The method is shown to converge under conditions which in some cases allow for the occurrence of a bound state. In this latter case the potential is not unique. However, the authors extend a result of the reviewer to show that if the normalizing parameter of each bound state is known as well as the phase, then the potential is uniquely determined. [A reference is made at the end to an announcement of results of Gelfand and Levitan which solves the problem in full generality. The detailed account is given in Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 309–360 (1951); these Rev. **13**, 558.]

N. Levinson (Cambridge, Mass.).

Schubert, Hans. Über die Potentiale der auf dem Mantel eines Kreiszylinders ausgebreiteten einfachen und doppelten Belegung. Math. Nachr. **8**, 249–255 (1952).

The author derives a Fourier integral representation containing Bessel functions for the axially symmetric potential induced by a simple and double layer on the surface of a circular cylinder.

D. Gilbarg (Bloomington, Ind.).

Differential Equations

4 ★**Gonzales, Mario O. Problems of differential equations.** Symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Diciembre, 1951, pp. 85–89. Centro de Cooperación Científica de la Unesco para América Latina, Montevideo, Uruguay, 1952. (Spanish)

Frei, M. Sur l'ordre des solutions entières d'une équation différentielle linéaire. C. R. Acad. Sci. Paris **236**, 38–40 (1953).

Using Nevanlinna's theory of meromorphic functions the author proves the following theorem. The general solution of the differential equation

$$w^{(n)}+a_{n-1}(z)w^{(n-1)}+\cdots+a_0(z)w=0,$$

where the a_k are entire functions, is an entire function of infinite order if at least one of the a_k is transcendental. If a_p is the transcendental coefficient with largest index p, then the equation possesses at most p independent solutions which are entire functions of finite order. (For a previous application of essentially the same method see Wittich [Math. Z. **47**, 422–426 (1941); these Rev. **4**, 42].)

W. H. J. Fuchs (Ithaca, N. Y.).

Hahn, Wolfgang. Über lineare Differentialgleichungen, deren Lösungen einer Rekursionsformel genügen. II. Math. Nachr. **7**, 85–104 (1952).

Continuing an earlier investigation [Math. Nachr. **4**, 1–11 (1951); these Rev. **13**, 233] the author shows the existence of polynomial chains (see the earlier review) which satisfy a fourth order linear homogeneous differential equation, and no equation of lower order. This is obtained as a special case of investigations which relate solutions of certain difference equations, approximating numerators and denominators of certain continued fractions, and polynomial chains. Two new methods of constructing new chains from given ones are discussed. An error in §4 of the earlier paper is corrected.

E. A. Coddington (Los Angeles, Calif.).

Russo, Salvatore. Sui sistemi di equazioni differenziali lineari, omogenei, a matrice costante e periodica. Boll. Un. Mat. Ital. (3) **7**, 428–430 (1952).

This note is concerned with a system of differential equations

$$y_i'=\sum_{j=1}^{n}a_{ij}y_j \quad (i=1, \cdots, n)$$

where the constant matrix $A=(a_{ij})$ satisfies the relation $A^p=I$ (p is a positive integer, and I the unit matrix). A few simple properties of the solutions are pointed out.

L. A. MacColl (New York, N. Y.).

Hinds, A. K., and Whyburn, W. M. A non-self-adjoint differential system of the second order. J. Elisha Mitchell Sci. Soc. **68**, 32–43 (1952).

The authors study the differential system

$$y'(x,\lambda)=K(x,\lambda)z(x,\lambda), \quad z'(x,\lambda)=G(x,\lambda)y(x,\lambda);$$
$$\psi(a,\lambda)=0, \qquad \varphi(a,\lambda)=\varphi(b,\lambda),$$

where

$$\psi(x,\lambda)=\gamma(x,\lambda)z(x,\lambda)-\delta(x,\lambda)y(x,\lambda),$$
$$\varphi(x,\lambda)=\alpha(x,\lambda)z(x,\lambda)-\beta(x,\lambda)y(x,\lambda).$$

A number of "eigentheorems" are obtained.

W. Leighton (St. Louis, Mo.).

Miller, Kenneth S. Construction of the Green's function of a linear differential system. Math. Mag. **26**, 1–8 (1952).

The author first explains, in more elementary style than previously [J. Appl. Phys. **22**, 1054–1057 (1951); these Rev. **13**, 348], the construction of the "one-sided Green's function" for an nth order ordinary linear differential equation, and gives a connection between it and the Green's function for two-point boundary conditions. Full details are given only for the second-order case.

F. V. Atkinson.

Pollaczek, Félix. Sur une application de l'intégrale d'Hadamard à la théorie des équations différentielles linéaires. C. R. Acad. Sci. Paris **235**, 681–684 (1952).

In the differential equation $L[y]=\sum_{i=0}^{n}q_i(x)x^{-i}y^{(n-i)}=0$ the coefficients $q_i(x)$ are supposed to be holomorphic at $x=0$. (The factor x^{-i} is erroneously missing in the paper.) If this differential equation has no integral characteristic exponents at $x=0$, and if $h(x)$ is holomorphic at $x=0$, then the differential equation $L[y]=x^{-n}h(x)$ possesses a unique solution $g(x)$ that is holomorphic at $x=0$. It is shown that this solution is the finite part, in the sense of Hadamard [see Le problème de Cauchy . . . , Hermann, Paris, 1932, pp. 184–217], of the integral

$$\int_0^x \begin{vmatrix} y_1(\xi)\cdots y_1^{(n-2)}(\xi)y_1(x) \\ \cdots \\ y_n(\xi)\cdots y_n^{(n-2)}(\xi)y_n(x)\end{vmatrix} \cdot \begin{vmatrix} y_1(\xi)\cdots y_1^{(n-1)}(\xi) \\ \cdots \\ y_n(\xi)\cdots y_n^{(n-1)}(\xi)\end{vmatrix}^{-1} \xi^{-n}h(\xi)d\xi.$$

W. R. Wasow (Los Angeles, Calif.).

Fuller, F. B. Note on trajectories in a solid torus. Ann. of Math. (2) **56**, 438–439 (1952).

The author outlines the construction of two "flows" of interest in the topological studies of systems of ordinary differential equations. The first exhibits a vector field (non-vanishing) in a solid (3-dimensional) torus with no closed integral curve (periodic solution) not contractible to a point in the torus. This example has periodic solutions and so leaves open the possibility that some version of the Poincaré-Bendixson Theorem carries over to 3 dimensions. In 4 dimensions, however, he exhibits a non-vanishing vector

field in a solid 4-dimensional torus with no closed integral curves (on p. 439 the author clearly intends $2'$ to be inside $3'$). *S. P. Diliberto* (Berkeley, Calif.).

DeBaggis, H. F. Dynamical systems with stable structures. Contributions to the Theory of Nonlinear Oscillations, vol. II, pp. 37–59. Princeton University Press, Princeton, 1952.

The system $dx_i/dt=P_i(x_1, x_2)$, $i=1, 2$, is considered in the phase plane. P_i are of class C_1. The author considers the requirement that small perturbations should not change the essential features of the system. He shows that critical points can be only nodes, foci, and saddle points; that a separatrix cannot both issue and terminate in saddle points, and that there are only a finite number of periodic solutions and the first variation for each of these has one characteristic exponent with real part not zero. Conversely these criteria suffice to insure structural stability of the configuration in the phase plane. *N. Levinson* (Cambridge, Mass.).

Lefschetz, Solomon. Notes on differential equations. Contributions to the Theory of Nonlinear Oscillations, vol. II, pp. 61–73. Princeton University Press, Princeton, 1952.

Part I: The topological structure of trajectories of an analytical system $\dot{x}=P(x, y)$, $\dot{y}=Q(x, y)$ near a critical point $(P=Q=0)$ is analyzed by elementary geometrical methods. The reviewer observes that similar methods have already been used, e.g., by E. R. Lonn [Math. Z. **44**, 507–530 (1938)]. Part II: The topological structure in the large of the trajectories of $\dot{x}=y-\lambda(x^3/3-x)$, $\dot{y}=x$ (equivalent to van der Pol's equation $\ddot{x}+\lambda(x^2-1)\dot{x}+x=0$) is completely described, by using the methods of Poincaré. The discussion includes a proof that every trajectory except $x=y=0$ tends to the unique limit cycle as $t\to\infty$. *G. E. H. Reuter.*

McCarthy, John. A method for the calculation of limit cycles by successive approximation. Contributions to the Theory of Nonlinear Oscillations, vol. II, pp. 75–79. Princeton University Press, Princeton, 1952.

Let (1) $\dot{x}=f(x)$, $x=(x_1, \cdots, x_n)$ have a solution $x=\phi(t)$ of period ω. Introduce new coordinates $(y_1, \cdots, y_{n-1}, s)$ near $x=\phi(t)$ which transform (1) into (2) $dy/ds=g(y, s)$, where g has period ω in s and $\phi(t)$ becomes $\psi(s)$ (with period ω). Let $G(y, s)=(\partial g_i/\partial y_j)$, and suppose $\psi(s)$ stable so that the characteristic exponents of $du/ds=G(\psi(s), s)u$ have negative real part. Starting with a function $\gamma_0(s)$ of period ω, define $\gamma_1(s), \gamma_2(s), \cdots$ inductively, $y=\gamma_{n+1}(s)$ being the unique solution with period ω of

$$dy/ds=g(\gamma_n(s), s)+G(\gamma_n(s), s)(y-\gamma_n(s)).$$

It is then shown that if $\gamma_0(s)$ is sufficiently near to $\psi(s)$, the sequence $\gamma_n(s)$ converges uniformly to $\psi(s)$. The reviewer cannot see how such a process could be used to calculate $\phi(t)$, since the process cannot be set up until $\phi(t)$ is known. *G. E. H. Reuter* (Manchester).

Taam, Choy-tak. Non-oscillatory differential equations. Duke Math. J. **19**, 493–497 (1952).

Call (1) $(p(x)y')'+f(x)y=0$ $(p(x)>0, x>0)$ disconjugate on $(0, \infty)$ if every solution $\not\equiv 0$ vanishes at most once on $(0, \infty)$, and nonoscillatory if it is disconjugate on (a, ∞) for some $a\geqq 0$. A criterion of Wintner [Amer. J. Math. **73**, 368–380 (1951); these Rev. **13**, 37] for (1) to be disconjugate is slightly extended and used to prove: If

$$(2)\qquad (P(x)y')'+F(x)y=0$$

is nonoscillatory,

$$\left|\int_x^\infty f(x)dx\right|\leqq\int_x^\infty F(x)dx,\quad p(x)\geqq P(x),\quad P(x)\leqq A$$

for $x>a$, then (1) is also nonoscillatory. For $p(x)=P(x)=1$, $f(x)$ and $F(x)\geqq 0$, this was proved by Hille [Trans. Amer. Math. Soc. **64**, 234–252 (1948), Theorem 7; these Rev. **10**, 376]. Another result of Hille [loc. cit., Theorem 13] is extended as follows: (1) is nonoscillatory if $p(x)\geqq 1$ and $|\int_x^\infty f(x)dx|\leqq\frac{1}{4}\int_x^\infty S_k(x)dx$ for $x>a$, where

$$S_k(x)=x^{-2}+(x\log x)^{-2}+(x\log x\log\log x)^{-2}+\cdots$$

to $k+1$ terms. *G. E. H. Reuter* (Manchester).

Hille, Einar. Behavior of solutions of linear second order differential equations. Ark. Mat. **2**, 25–41 (1952).

The author considers the differential equation

$$w''=\lambda F(x)w,$$

where $F(x)$ is positive and continuous on the interval I: $0\leqq x<\infty$, and λ is a complex parameter which, in general, is not permitted to assume real negative values. It is proved that if $G(x)$ is continuous on I and if there exists a real number β and a positive number δ such that $|\arg[e^{-i\beta}G(x)]|\leqq\frac{1}{2}\pi-\delta$ for all large x, the differential equation $w''=G(x)w$ possesses a fundamental system of solutions of the form $w_1(x)=x[1+o(1)]$, $w_2(x)=1+o(1)$ as $x\to\infty$ if and only if $xG(x)\in L(0, \infty)$.

After proving the last result, the author assumes in the remainder of the paper that $xF(x)$ non-$\in L(0, \infty)$ with λ restricted as indicated above. A typical theorem is the following. If $xF(x)$ non-$\in L(0, \infty)$, if $\lambda=\mu+i\nu\in\Lambda$ and $\nu\neq 0$, then $w_k(x, \lambda)$ describes a spiral $S_k(\lambda)$ from k to ∞ in the complex w-plane as x goes from 0 to $+\infty$, $k=0, 1$, and $\nu^{-1}\arg w_k(x, \lambda)$ increases steadily from 0 to $+\infty$ with x. $S_k(\lambda)$ has a positive radius of curvature everywhere and is concave towards the origin. If $\mu\geqq 0$, $|w_k(x, \lambda)|$ is monotone increasing and $|w_k(x, \lambda)|/x\to\infty$ with x when $\mu>0$. For all $\lambda\in\Lambda$, $|w_k(x, \lambda)|^{-1}\in L_2(1, \infty)$. The paper concludes with results on the rate of growth of solutions. *W. Leighton.*

Munakata, Ken-iti. Some exact solutions in nonlinear oscillations. J. Phys. Soc. Japan **7**, 383–391 (1952).

Solutions of Duffing's equation $\ddot{x}+\alpha x+\beta x^3=0$ are expressed in terms of Jacobian elliptic functions: cn if $\alpha>0$, $\beta>0$; sn if $\alpha>0$, $\beta<0$; dn if $\alpha<0$, $\beta>0$. It is then shown that if $\alpha>0$, $\beta>0$, $\ddot{x}+\alpha x+\beta x^3=F$ cn $(4K\omega t, k)$ has one or more periodic solutions $x=A$ cn $(4K\omega t, k)$; that is, if F and ω are given, one or more values of A and k can be determined. Results are plotted as response curves of A against ω for fixed F. When $\alpha>0$, $\beta<0$, similar results hold with cn replaced by sn. Stability is investigated by solving the variational equation explicitly. The author's assertion that his results contradict those of F. John [Communications on Appl. Math. **1**, 341–359 (1948); these Rev. **10**, 709] appears unjustified to the reviewer. John considers an equation whose forcing term has a wave form independent of its period, whilst here the form of F cn $(4K\omega t, k)$ depends on k and hence on ω. *G. E. H. Reuter* (Manchester).

Colombo, Giuseppe. Sopra un sistema non-lineare in due gradi di libertà. Rend. Sem. Mat. Univ. Padova **21**, 64–98 (1952).

For constant $\alpha, \beta, \gamma, \delta, \rho, m_1$ and m_2 and small ϵ the author investigates the existence of periodic solutions, of period

near 2π, for the system

$$\ddot{x}+x=\epsilon(\alpha\dot{x}-3\gamma x^2\dot{x}-m_1y),\quad \ddot{y}+y=\epsilon(\beta\dot{y}-3\delta y^2\dot{y}-m_2x-\rho y).$$

The author adopts the method of Cartwright and Littlewood [Cartwright, Contributions to the theory of nonlinear oscillations, v. I, Princeton, 1950, pp. 149–241, esp. p. 202 ff; these Rev. 11, 722]. Stability is also considered.

N. Levinson (Cambridge, Mass.).

Conti, Roberto. Soluzioni periodiche dell'equazione di Liénard generalizzata. Esistenza ed unicità. Boll. Un. Mat. Ital. (3) **7**, 111–118 (1952).

A theorem demonstrating the existence of a periodic solution for $x''+f(x)x'+g(x)=0$ by Filippov [Mat. Sbornik N. S. **30**(72), 171–180 (1952); these Rev. **13**, 944] is shown to include Theorem I of Levinson and O. K. Smith [Duke Math. J. **9**, 382–403 (1942); these Rev. **4**, 42]. A uniqueness criterion for the periodic solution is also given.

N. Levinson (Cambridge, Mass.).

Schaffner, Johannes S. Almost sinusoidal oscillations in nonlinear systems. II. Synchronization. University of Illinois Bulletin. Engineering Experiment Station Bulletin Series, no. **400**, 32 pp. (1952).

For part I see same Bulletin Series, no. **395** (1951); these Rev. **13**, 238.

Roberson, Robert E. On the relationship between the Martiensson and Duffing methods for nonlinear vibrations. Quart. Appl. Math. **10**, 270–272 (1952).

Thomas, J. M. Equations equivalent to a linear differential equation. Proc. Amer. Math. Soc. **3**, 899–903 (1952).

It is shown that the differential equations

$$y'+p(x)f(y)\int\frac{dy}{f(y)}+q(x)f(y)=0,$$

$$y''+p(x)y'+q(x)g(y)=[g'(y)-1][g(y)]^{-1}y'^2$$

can be solved by setting $y=F(u)$, where in the first case u satisfies the equation $u'+pu+q=0$ and F is determined by the equation $F'=f(F)$, and in the second case u satisfies the equation $u''+pu'+qu=0$ and F is determined by the equation $uF'=g(F)$. A few other cases, in which differential equations can be solved by means of similar but more complicated methods, are discussed also. *L. A. MacColl.*

Klamkin, M. S. Generalization of Clairaut's differential equation and the analogous difference equation. Amer. Math. Monthly **60**, 97–99 (1953).

Visvanathan, S. On the use of auxiliary differential equations in orthogonal expansions. Math. Student **20**, 58–62 (1952).

Let $L[u]$ denote a self-adjoint differential operator $[R(x)u'(\nu,x)]'+P(\nu,x)u(\nu,x)$ such that the characteristic numbers ν and functions $u(\nu,x)$ of an eigenvalue problem in $L[u]=0$ are known. The Lagrange identity for $L[u]$ is used to point out that when a function $f(x)$ is such that a particular solution $G(\nu,x)$ of the differential equation $L[G]=f$ is known, then Fourier constants of the type $C_\nu=\int_a^b f(x)u(\nu,x)dx$ can be evaluated easily by means of the formula $C_\nu=[(uG'-Gu')R]_a^b$. Examples are given for particular functions $f(x)$ in the case of ordinary Fourier series and integrals. *R. V. Churchill* (Ann Arbor, Mich.).

Krein, M. G. On the indeterminate case of the Sturm-Liouville boundary problem in the interval $(0,\infty)$. Izvestiya Akad. Nauk SSSR. Ser. Mat. **16**, 293–324 (1952). (Russian)

The differential equation (*) $y''+q(x)y+\lambda\rho(x)y=0$ is considered over $0\leqq x<\infty$. It is assumed to be in the limit circle case. The functions q and ρ are real and integrable on every finite interval and $\int\rho(x)dx>0$ over any open interval (a,b). Let $\phi(x)$ and $\psi(x)$ be two linearly independent solutions for the case $\lambda=0$ and consider the boundary conditions $W(y,\phi)|_{x=0}=0$, $\lim_{x\to\infty}W(y,\psi)=0$ for (*) where $W(y,z)=yz'-y'z$. Then the spectrum consists of the real sequence λ_j. The author shows that the λ_j are the zeros of an entire function $D(\lambda)$ where

$$\limsup \log|D(\lambda)|/|\lambda|=0 \quad\text{as}\quad |\lambda|\to\infty$$

and

$$\frac{1}{D(\lambda)}=1+\lambda\sum_{j=1}^{\infty}\frac{1}{\lambda_jD'(\lambda_j)(\lambda-\lambda_j)}.$$

If $n_+(r)$ denotes the number of λ_j in $(0,r)$ and if

$$S=\sum_{\lambda_j<0}1/|\lambda_j|^{1/2}<\infty,$$

then as $r\to\infty$, $\lim n_+(r)/r^{1/2}=\int_0^\infty\rho^{1/2}(x)dx<\infty$. However, if the integral of $\rho^{1/2}$ diverges, then $S=\infty$ and $\lim n_+(r)/r^{1/2}=\infty$ as $r\to\infty$. *N. Levinson* (Cambridge, Mass.).

Hartman, Philip. Some examples in the theory of singular boundary value problems. Amer. J. Math. **74**, 107–126 (1952).

Let q be a real-valued continuous function on $0\leqq t<\infty$, and λ a real parameter. Let $N(T,\lambda)$ denote the number of zeros on $0<t<T$ of a non-trivial solution of $x''+(q+\lambda)x=0$. Various examples are given to show how pathological $N(T,\lambda)$ may be. These examples are then interpreted in terms of the spectral theory for the given differential equation. [Gelfand and Levitan [Doklady Akad. Nauk SSSR (N.S.) **77**, 557–560 (1951); Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 309–360 (1951); these Rev. **13**, 240, 558] have shown that under rather mild restrictions on a monotone non-decreasing function ρ on $-\infty<\lambda<+\infty$, there exists a continuous q on $0\leqq t<\infty$ such that $x''+(q+\lambda)x=0$ and a homogeneous boundary condition at zero have ρ as its spectral function.] *E. A. Coddington.*

Fishel, B. On two papers of Titchmarsh concerning eigenfunction expansions for partial differential operators of elliptic type. J. London Math. Soc. **27**, 496–502 (1952).

This paper deals with expansion theorems associated with the boundary problem consisting of the differential equation

(*) $\nabla^2u(x,y)+\{\lambda-q(x,y)\}u(x,y)=0,\quad (x,y)\in G,$

and either the boundary condition (i) $u=0$ or

(ii) $u+p\partial u/\partial n=0$

on the boundary of G, where G is a simply connected and bounded region in the (x,y)-plane whose boundary has a continuously turning tangent. The method of proof employed is an adaptation of argument used by Titchmarsh [Proc. London Math. Soc. (3) **1**, 1–27 (1951); these Rev. **13**, 241] for a related boundary problem involving (*).

W. T. Reid (Evanston, Ill.).

★**Saltikov, N. Teorija parcijalnih jednačina drugog reda. [The theory of partial differential equations of the second order.]** Naučna Knjiga, Belgrade, 1952. 121 pp.

Ce livre reproduisant l'enseignement de l'auteur à l'Université de Belgrade expose les progrès acquis dans le domaine de la théorie des équations aux dérivées partielles du second ordre. Signalant l'importance du sujet exposé pour l'évolution des mathématiques modernes, l'auteur constate la distinction qui existe entre les théories des équations aux dérivées partielles de premier et celles de second ordre.

Après avoir donné les notions et les définitions fondamentales des équations étudiées, au premier chapitre, le second traite sur les méthodes immédiates de leur intégration. Le but est de présenter les équations étudiées sous une forme qui se prête à l'intégration immédiate, sans utiliser les méthodes spéciales d'intégration exigeant parfois des opérations encombrantes. De telle manière sont intégrées toutes, les équations des cours classiques d'analyse auxquelles on y applique les méthodes d'intégration qui sont souvent trop compliquées. Les chapitres trois et quatre, exposent à titre d'évolution des méthodes citées les problèmes d'intégration des équations de Laplace et des équations linéaires de la forme générale. Les suites classiques des invariants sont supplées par les suites de ceux qui appartiennent à la même classe d'ordres inférieurs. Cela simplifie la théorie d'intégrations en prêtant à elle plus d'élégence, et facilite en même temps les applications.

Le cinquième chapitre étudie de différentes intégrales des équations aux dérivées partielles du second ordre. Constatant certaines notions erronées d'intégrales que l'on rencontre même chez des géométres les plus éminants, comme Bertrand et Forsyth, pour s'en émanciper, l'auteur introduit la notions des intégrales générales mixtes. Le suivant chapitre donne une étude analytique du problème d'intégration des équations de Monge-Ampère. Le chapitre sept traite des transformations de contact et de leur application à l'intégration des équations étudiées du second ordre.

Au chapitre huit est exposé la démonstration de l'existence de l'intégrale de Cauchy, en élargissant considérablement le domaine de leur existence. C'est ainsi qu'une équation linéaire par rapport aux dérivées partialles des deux premiers ordres et à la fonction inconnue admet l'intégrale de Cauchy dans tout le domaine de holomorphie des coefficients des dites équations.

Le dernier chapitre étudie les systèmes d'équations du second ordre. L'auteur insiste sur les deux genres de leur involution: d'intégrabilité complète et celle de Darboux-S. Lie, en signalant leur distinction et leurs avantages. La nouvelle méthode d'intégration des équations du second ordre, y est exposée, généralisant la première méthode Lagrangienne concernant les équations aux dérivées partielles du premier ordre.

Ce travail traitant les équations aux dérivées partielles du second ordre est conçu de la manière pour servir d'introduction à la théorie générale des équations aux dérivées partielles d'ordres supérieurs au second. *K. Orloff.*

★**Krilov, A. N. O nekim diferencijalnim jednačinama tehničke fizike. [On some differential equations of technical physics.]** Naučna Knjiga, Belgrade, 1952. xii+431 pp.

Translated from Krylov's "O nekotoryh differencial'nyh uravneniyah matematičeskoĭ fiziki imeyuščih priloženie v tehničeskih voprosah", 3d ed. [Leningrad, 1933; cf. these Rev. 12, 615].

★**Sauer, Robert. Anfangswertprobleme bei partiellen Differentialgleichungen.** Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete. Band LXII. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1952. xiv+229 pp. 26 DM; Bound, 29 DM.

There are two features of the book that should make it useful for physicists and engineers. The author gives an abundance of examples to illustrate the theories developed, mostly taken from gas dynamics, though also some problems arising in differential geometry and elasticity are discussed. Secondly, numerical methods are emphasized throughout the book, indicating how solutions can be computed in practice.

The major part of the book is devoted to hyperbolic equations in two independent variables. Chapter I deals with the different behavior of solutions of elliptic, parabolic, and hyperbolic equations and the problems appropriate to each type. Equations with constant coefficients are used to introduce the notions of characteristics, propagation of discontinuities, and domains of dependence. Approximation of solutions by power series and by finite differences are discussed. Chapter II contains an exposition of the standard theory of integration of a single first-order equation and its equivalence with systems of ordinary differential equations. Connections with Riemannian geometry and with the Hamilton-Jacobi theory are discussed. Chapter III gives an up-to-date account of the theory of systems of first-order equations and single second-order equations in two independent variables, reflecting the progress in this field made by Cinquini-Cibrario, Courant, Friedrichs, Lax, Lewy, and others in recent years. Quasi-linear systems of two first order equations are referred to characteristic parameters, and the resulting systems are solved by iteration and by the method of finite differences, with proofs of convergence given. The analogous theory for higher-order systems is given without proof. The chapter closes with Riemann's method of integration, giving special cases, where the Riemann function can be expressed in terms of hypergeometric functions. Chapter IV brings equations in more than two independent variables, mostly linear equations of second order. Characteristic surfaces, discontinuous solutions, explicit solutions for equations with constant coefficients, principle of descent. The last 30 pages of the book are devoted to a rather complete exposition of Hadamard's theory of integration of the general linear hyperbolic equations of second order. *F. John* (New York, N. Y.).

Cooperman, Philip. An extension of the method of Trefftz for finding local bounds on the solutions of boundary value problems, and on their derivatives. Quart. Appl. Math. 10, 359–373 (1953).

Zunächst wird die Methode von Friedrichs [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. **1929**, 13–20] zur Auffindung von oberen und unteren Schranken für das Dirichletsche Integral $\int(u_x^2+u_y^2)df$, wobei u der Poissonsche Differentialgleichung genügt, auf den Fall erweitert, dass für einen Teil des Randes u und für einen zweiten Teil des Randes $\partial u/\partial n$ gegeben ist. Um Aussagen für obere und untere Schranken für die Werte der Funktion $u=u(x, y)$ selbst zu gewinnen, wird die Darstellung von u mit Hilfe der Greenschen Funktion herangezogen. Zerlegt man die Greensche Funktion in

$$G=\frac{1}{2\pi}\ln r+R,\quad r=((x-\xi)^2+(y-\eta)^2)^{1/2},$$

wobei R ein reguläres logarithmisches Potential ist, so wird die Methode von Friedrichs zur Gewinnung von oberen und unteren Schranken auf Dirichletsche Integrale über die Funktion R und schliesslich auch auf solche über eine Funktion von der Form $q=u-\alpha R$ angewandt. Die Diskriminantenbedingung für die so entstehenden, in α quadratischen Ungleichungen liefert die gewünschte obere und untere Schranke für die Werte von $u(x,y)$. Dieselbe Methode lässt sich auch heranziehen, um u_x und u_y abzuschätzen.

Der Verfasser verallgemeinert sein Verfahren auf Systeme von mehreren partiellen Differentialgleichungen zweiter Ordnung für eine beliebige Anzahl von unabhängigen Veränderlichen, die aus analogen Variationsproblemen entspringen. Es folgen Anwendungen auf die Randwertprobleme der Elastizitätstheorie und auf das Randwertproblem bei einer beliebigen sich selbst adjungierten partiellen Differentialgleichung zweiter Ordnung. *P. Funk.*

Oleĭnik, O. A. On boundary problems for equations with a small parameter in the highest derivatives. Doklady Akad. Nauk SSSR (N.S.) **85**, 493–495 (1952). (Russian)

The elliptic equation considered by the author [same Doklady (N.S.) **79**, 735–737 (1951); these Rev. **13**, 559] is now considered with boundary condition $\partial u/\partial n+au=\phi$. It is assumed that the function $a\leqq 0$ and $c(x,y)<0$. The parabolic equation (in which the term u_{yy} is omitted) is also discussed. *N. Levinson* (Cambridge, Mass.).

Oleĭnik, O. A. On equations of elliptic type with a small parameter in the highest derivatives. Mat. Sbornik N.S. **31**(73), 104–117 (1952). (Russian)

Proofs are given for the results stated in Doklady Akad. Nauk SSSR (N.S.) **79**, 735–737 (1951) [these Rev. **13**, 559] and the paper reviewed above. *N. Levinson.*

Wassermann, G. D. Heat conduction in solids as an eigenvalue problem. Quart. J. Mech. Appl. Math. **5**, 466–471 (1952).

The author begins with formal relations pointed out by A. G. Walters [Proc. Cambridge Philos. Soc. **45**, 69–80 (1949); these Rev. **10**, 196] between the Green's functions for the heat equation $V_t=k\nabla^2V$ and for the equation $\nabla^2G=\lambda^2G$. The bilinear formula is used formally to express the first Green's function V in a series involving the characteristic functions of an eigenvalue problem in the equation $\nabla^2G=\lambda^2G$. The usual integrals involving V then represent general solutions of transient heat conduction problems. Examples are given. *R. V. Churchill.*

Voskresenskiĭ, K. D. On a nonlinear problem of the theory of heat conduction. Doklady Akad. Nauk SSSR (N.S.) **87**, 575–576 (1952). (Russian)

An isotropic body, without interior heat sources, is considered. A piecewise constant distribution of temperature is given on the surface of the body. The stationary temperature field in the body is described by the nonlinear differential equation div $[\lambda(t)\,\mathrm{grad}\,t]=0$, where $\lambda(t)$ is the given coefficient of heat conductivity. It is shown how this nonlinear problem can be reduced to an analogous linear problem with a constant coefficient of heat conductivity.
H. P. Thielman (Ames, Iowa).

Gol'dfarb, È. M. Application of the method of sources for solution of the heat conduction equation. Akad. Nauk SSSR. Žurnal Tehn. Fis. **22**, 1606–1617 (1952). (Russian)

The essence of the method of sources consists in the representation of the heat distribution within a body as the result of the equalization of the temperature produced by a set of elementary heat quantities (sources) distributed in space as well as in time. The solution of a heat conduction problem by this method consists in the correct distribution of these sources. The present paper describes the proper arrangement of sources for the temperature distribution within infinite slabs under various initial and boundary conditions. A number of particular examples are worked out in detail. *H. P. Thielman* (Ames, Iowa).

Dacev, A. B. On the appearance of a phase in the linear problem of Stefan. Doklady Akad. Nauk SSSR (N.S.) **87**, 353–356 (1952). (Russian)

The linear problem of Stefan for a rod is considered with a set of initial conditions such that there is a single phase present at the time $t=0$ between two specified points of the rod. Under suitable conditions it is shown that the other phase will appear and that the temperature within each of the two phases may be given in terms of solutions developed in preceding papers [principally, Dacev, same Doklady (N.S.) **58**, 563–566 (1947); **74**, 445–448 (1950); these Rev. **9**, 513; **12**, 263] if the position of the boundary of separation follows an assumed parabolic law. The method consists of the consideration of two auxiliary problems and showing that the functions describing the two phases can be expressed as a linear combination of the solutions of the two auxiliary problems. *C. G. Maple* (Ames, Iowa).

Yosida, Kôsaku. On the integration of diffusion equations in Riemannian spaces. Proc. Amer. Math. Soc. **3**, 864–873 (1952).

Let R be a connected domain of an infinitely differentiable orientable Riemannian space of dimension $m\geqq 2$ with the metric $ds^2=g_{ij}(x)dx^idx^j$. The author considers the equations

$$(1)\quad \frac{\partial f(x,t)}{\partial t}=b^{ij}(x)\frac{\partial^2 f(x,t)}{\partial x^i\partial x^j}+a^i(x)\frac{\partial f(x,t)}{\partial x^i},\qquad t\geqq 0,$$

$$(2)\quad \frac{\partial h(x,t)}{\partial t}=[g(x)]^{-1/2}\left\{\frac{\partial^2}{\partial x^i\partial x^j}([g(x)]^{1/2}b^{ij}(x)h(x,t)) -\frac{\partial}{\partial x^i}([g(x)]^{1/2}a^i(x)h(x,t))\right\},\quad t\geqq 0.$$

Here $g(x)=\det[g_{ij}(x)]$, the symmetric contravariant tensor $b^{ij}(x)$ makes $b^{ij}(x)\xi_i\xi_j$ into a positive definite quadratic form for $x\in R$, and the $a^i(x)$ obey a rule of coordinate transformation making the two elliptic differential operators in the right members formally adjoint to each other. The coefficients are in C^∞ with respect to the local coordinates. The author proves, if R is a compact space, that to any real $f(x)\in C^\infty$ in R, there is a unique solution $f(x,t)$ of (1) tending uniformly to $f(x)$ when $t\to 0$ and lying between min $f(x)$ and max $f(x)$ with max $f(x,t)=$ max $f(x)$ if $f(x)\geqq 0$. Likewise, to any given $h(x)\in C^\infty$ there is a unique solution $h(x,t)$ of (2) converging in the mean of order one (with respect to $dx=[g(x)]^{\frac{1}{2}}dx_1\cdots dx_m$) as $t\to 0$ such that

$$\int_R |h(x,t)|\,dx\leqq \int_R |h(x)|\,dx$$

with equality when $h(x)\geqq 0$, in which case also $h(x,t)\geqq 0$. Finally, if R is merely a connected domain with smooth boundary ∂R, it is shown that to every $h(x)\in C^\infty$ with compact support, satisfying a certain boundary condition on ∂R, there is a solution $h(x,t)$ with properties as above if and only if the first differential operator does not have any

positive characteristic values with characteristic functions satisfying a certain limiting orthogonality condition on ∂R. The proofs are based on the author's previous work [Ark. Mat. 1, 71–75 (1949); J. Math. Soc. Japan 3, 69–73 (1951); Nagoya Math. J. 3, 1–4 (1951); these Rev. 11, 443; 13, 560] for the operator theoretical part (existence of certain semi-groups) plus an elegant construction of a suitable parametrix. *E. Hille* (Nancy).

Feller, William. On a generalization of Marcel Riesz' potentials and the semi-groups generated by them. Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 72–81 (1952).

The author introduces the generalized Riesz potentials

$$I_\delta^\alpha f=(\Gamma(\alpha)\sin\alpha\pi)^{-1}\int_{-\infty}^{\infty}f(y)|y-x|^{\alpha-1}\times\sin\alpha\left(\frac{\pi}{2}+\frac{y-x}{|y-x|}\delta\right)dy$$

for sufficiently regular f. Its connection with the Riemann-Liouville integrals J_+^α, J_-^α is given by

$$I_\delta^\alpha=\frac{\sin\alpha(\delta+\frac{1}{2}\pi)}{\sin\alpha\pi}J_+^\alpha+\frac{\sin\alpha(-\delta+\frac{1}{2}\pi)}{\sin\alpha\pi}J_-^\alpha,$$

$$I_0^\alpha f=I^\alpha f=\left(2\Gamma(\alpha)\cos\frac{\pi\alpha}{2}\right)^{-1}\int_{-\infty}^{\infty}f(y)|y-x|^{\alpha-1}dy,$$

$$I^\alpha_{\pi/2}=J_+^\alpha,\quad I^\alpha_{-\pi/2}=J_-^\alpha.$$

From the well-known properties of the latter [M. Riesz, Acta Math. 81, 1–223 (1948); these Rev. 10, 713], we have $I_\delta^0=I$, $I_\delta^{-2}=-d^2/dx^2$, $I_\delta^{\alpha+\beta}=I_\delta^\alpha I_\delta^\beta$. According to P. Lévy [Théorie de l'addition des variables aléatoires, Gauthier-Villars, Paris, 1937], the stable law is defined by the density function $K_{\alpha\gamma}$ whose Fourier transform is given by

$$\Phi(z)=\exp\left\{-\tau|z|^\alpha\left(1+i\gamma\frac{z}{|z|}\tan\frac{\pi\alpha}{2}\right)\right\}.$$

The semi-group T_t, $t>0$, defined by

$$u(t,x)=T_tf(x)=t^{-1/\alpha}\int_{-\infty}^{\infty}f(y)K_{\alpha\gamma}(t^{-1/\alpha}(x-y))dy$$

includes, as special cases, the integral transforms with Gauss kernel and Poisson kernel. The former corresponds to the classical diffusion equation $\partial u/\partial t=\partial^2u/\partial x^2$, $u(0,x)=f(x)$. The semi-group corresponding to the symmetrical density $K_{\alpha 0}(x)$ is generated by the infinitesimal transformation $-(-d^2/dx^2)^{\alpha/2}$. This is an operator-theoretical formulation of S. Bochner's discovery linking the stable distributions to the diffusion equations in extended sense [Proc. Nat. Acad. Sci. U. S. A. 35, 368–370 (1949); these Rev. 10, 720]. Thus it is natural to introduce the functional equation $\partial u(t,x)/\partial t=-I_\delta^{-\alpha}u(t,x)$. It is shown that, for $0<\alpha<1$, its formal solution $u(t,x)=\sum(-t)^nI_\delta^{-n}f/n!$ can be written as an integral transform of f with the kernel $U_{\alpha\delta}(x-y)$, where

$$(1)\quad U_{\alpha\delta}(x)=\frac{-1}{\pi|x|}\sum\left(\frac{-t}{|x|^\alpha}\right)^n\frac{\Gamma(1+n\alpha)}{n!}\sin n\alpha\left(\frac{\pi}{2}+\frac{x}{|x|}\delta\right).$$

It is proved that, when $0<\alpha<1$ and $\alpha\delta<\pi/2$, $U_{\alpha\delta}(x)$ exhausts stable density functions. By means of the formula

$$U_{\alpha\delta}(x)=\frac{1}{\pi}\Re\int_0^\infty\exp\left(-ir|x|-tr^\alpha\exp\left(-i\alpha\delta x/|x|\right)\right)dr$$

which is obtainable by Cauchy's integral formula from (1), the analytic continuation of (1) is possible and thus all stable densities with $1<\alpha\leqq 2$ are given by

$$(2)\quad U_{\alpha\delta}(x)=\frac{-1}{\pi|x|}\times\Im\sum\frac{(-i|x|\exp(i\delta x/|x|))^{n+1}}{(n+1)!}\Gamma\left(\frac{n+1}{\alpha}+1\right).$$

The case $\alpha=1$, $\delta=0$ is of special interest. Both series (1) and (2) break down but lead nevertheless to the Poisson kernel. This results from the formula

$$I^{-1}=I^{-2}I=-d^2/dx^2\cdot I=-d/dx\ \text{(Hilbert transform)}$$

and the properties of the conjugate harmonic functions. Therefore the formal solution $\sum(-t)^nI^{-n}f/n!$ of

$$\partial\Phi/\partial t=-I^{-1}\Phi$$

leads to the harmonic function in the upper half-plane, in accordance with Bochner's result referred to above. *K. Yosida* (Osaka).

Pleijel, Åke. Sur les valeurs et les fonctions propres des membranes vibrantes. Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 173–180 (1952).

The membrane eigenvalue problems considered consist of the differential equation

$$\left(\frac{\partial^2}{\partial x_1}\right)^2u+\left(\frac{\partial^2}{\partial x_2}\right)^2u+\lambda u=0$$

in a plane domain V, subject to one of the boundary conditions $u=0$, or $\partial u/\partial n=0$, on the boundary S of V. T. Carleman [Åttonde Skandinaviska Matematikerkongressen, Stockholm, 1934, pp. 34–44, Ohlsson, Lund, 1935] introduced the idea of reducing the problem of eigenvalue-eigenfunction distribution to an investigation of the behavior (finding estimates for) the compensating part (regular part) of the appropriate Green's function, and obtained a new proof of Weyl's asymptotic law for the eigenvalues λ_n. Carleman showed that the series

$$F(s)=\sum_{n=0}^{\infty}\lambda_n^{-s}$$

can be continued analytically to the half-plane $\mathrm{Re}(s)>\frac{1}{2}$, and obtained the formula

$$F(s)=\frac{V}{4\pi}\cdot\frac{1}{s-1}+\chi(s),$$

where V denotes the area of the domain V and $\chi(s)$ is analytic on $\mathrm{Re}(s)>\frac{1}{2}$. The present author (with T. Ganelius) has previously found estimates for the regular parts of Green's functions for the corresponding problems for the Laplace equation in three dimensions, and also for analogous problems for the biharmonic equation [Pleijel, Proc. Symposium on Spectral Theory and Differential Problems, Oklahoma Agricultural and Mechanical College, Stillwater, Okla., 1951, pp. 413–437; these Rev. 13, 948]. In the present paper, the author finds that these methods are, after suitable modifications, applicable to the present problems, and obtains the following improvement of Carleman's analytic continuation result quoted above:

$$\sum_{n=0}^{\infty}\lambda_n^{-s}=\frac{V}{4\pi}\cdot\frac{1}{s-1}\mp\frac{S}{8\pi}\cdot\frac{1}{s-\frac{1}{2}}+\chi(s),$$

for the first boundary condition, and

$$\sum_{n=1}^{\infty}\lambda_n^{-s}=\frac{V}{4\pi}\frac{1}{s-1}+\frac{S}{8\pi}\frac{1}{s-\frac{1}{2}}+\chi(s),$$

for the second boundary condition, where S is the length of the boundary of V, and $\chi(s)$ is analytic on $\mathrm{Re}(s)>-1$. It is shown that the Dirichlet series under consideration may be continued analytically for all s, and that the analytic continuation has simple poles at $s=1, 1/2, -1/2, -3/2, \cdots$. The author suggests that these results indicate the possibility of replacing Weyl's asymptotic law in both these eigenvalue problems by more detailed estimates.

J. B. Diaz (College Park, Md.).

Integral Equations

Hovanskiĭ, A. N. On a generalization of Abel's integral equation. Doklady Akad. Nauk SSSR (N.S.) **50**, 69–70 (1945). (Russian)

The author considers the generalized Abel integral equation

$$F(x)=\int_a^x\frac{\Phi(s)ds}{[\pm(\varphi(x)-\psi(s))]^{1/2}},$$

where $\varphi'(x)$, $\psi'(s)$ exist and do not vanish. He reduces it to an ordinary Abel equation by the substitutions $\varphi(x)=u$, $\psi(s)=v$, solves this, and reverses the substitutions to obtain the solution in the form

$$\Phi(s)=\pm\frac{1}{\pi}\frac{d}{ds}\int_a^s\frac{F(x)\varphi'(x)dx}{[\pm(\psi(s)-\varphi(x))]^{1/2}}.$$

F. Smithies (Nancy).

Colombo, Serge. Sur les équations intégrales de Volterra à noyaux logarithmiques. C. R. Acad. Sci. Paris **235**, 928–929 (1952).

Let $K(t)=t\log t+mt+n$, where m and n are constants and $n<0$. The integral equation $\int_0^t K(\xi)f(t-\xi)d\xi=g(t)$ is solved for $f(t)$ by using Laplace transforms. The author comments on the corresponding problem when $K(t)$ is a polynomial in $\log t$.

R. V. Churchill.

Germay, R. H. Sur les fonctions généralisant les noyaux itérés des systèmes d'équations intégrales de Volterra, de seconde espèce. II. Bull. Soc. Roy. Sci. Liège **21**, 42–45 (1952).

Conclusion of part I [same Bull. **21**, 2–6 (1952); these Rev. **14**, 53].

I. A. Barnett (Cincinnati, Ohio).

Parodi, Maurice. Application de la relation qui donne l'original d'un déterminant à la résolution d'un type d'équations intégrales. C. R. Acad. Sci. Paris **235**, 1002–1003 (1952).

Nikolenko, V. N. Cauchy's problem for an integro-differential equation of Fredholm type. Uspehi Matem. Nauk (N.S.) **7**, no. 5(51), 225–228 (1952). (Russian)

The author considers an integro-differential equation of the form

$$(1)\qquad L[y(x)]=\lambda\int_a^b\sum_{h=0}^{m}K_h(x,t)y^{(n-h)}(t)dt,$$

where $L[y(x)]\equiv y^{(n)}(x)+\lambda\sum_{k=1}^{n}a_k(x)y^{(n-k)}(x)$, the functions $a_k(x)$ are continuous, the kernels $K_h(x,t)$ are bounded and integrable, and $n\geqq m$. By putting

$$y(x)=\sum_{s=0}^{n-1}C_s\frac{(x-x_0)^s}{s!}+\int_{x_0}^{x}\frac{(x-\tau)^{n-1}}{(n-1)!}z(\tau)d\tau,$$

where the C_s are arbitrary constants, and $a\leqq x_0\leqq b$, he obtains a Fredholm integral equation of the form

$$(2)\qquad z(x)=\sum_{s=0}^{n-1}C_sA_s(x)+\lambda\int_a^b K(x,x_0,\tau)z(\tau)d\tau$$

for $z(x)$. He calls x_0 a regular point if λ is not a characteristic value of (2), and he proves the following results.

(i) If x_0 is a regular point, (1) has a unique solution with given values of $y, y', \cdots, y^{(n-1)}$ at x_0; in other words, the Cauchy problem is soluble for x_0. (ii) If there is at least one regular point in $a\leqq x\leqq b$, (1) has n linearly independent solutions. (iii) If λ is a characteristic value of (2) of rank r, with characteristic functions $z_1(x), \cdots, z_r(x)$, and the matrix $[(A_s, z_k)]$ has rank q, then (1) has $n+r-q$ linearly independent solutions. (iv) In case (iii), $n-q$ initial values can be prescribed at x_0, and the solution of the corresponding problem contains r arbitrary constants.

F. Smithies.

Nardini, Renato. Sull'unicità della soluzione di un'equazione integro-differenziale della fisica-matematica. Ann. Univ. Ferrara. Parte I. **8** (1948–50), 41–47 (1951).

The integro-differential equation is

$$u_{tt}+a^2u_{xxxx}+\int_0^t\Phi(t-\tau)u_{xxxx}(x,\tau)d\tau=F(x,t).$$

u and u_{xx} vanish when $x=0$ and $x=l$, and u, u_t are prescribed when $x=0$. The author uses the Laplace transformation to prove the uniqueness of the solution under certain conditions (among them the condition that it be permissible to interchange the Laplace transformation with respect to t, and differentiation with respect to x).

A. Erdélyi.

Dörr, Johannes. Zwei Integralgleichungen erster Art, die sich mit Hilfe Mathieuscher Funktionen lösen lassen. Z. Angew. Math. Physik **3**, 427–439 (1952).

The author considers the following integral equations:

$$g(x)=\int_{-1}^{1}h(y)Z_0(k|x-y|)dy,$$

$$w(x)=k\int_{-1}^{1}p(y)\frac{|x-y|}{x-y}Z_1(k|x-y|)dy,$$

where $Z_n(\alpha)=AJ_n(\alpha)+BN_n(\alpha)$, A and B arbitrary complex constants, J_n and N_n Bessel and Neumann functions of nth order. Results for the second equation follow easily from those for the first. It is shown that, after a change of variables, the even Mathieu functions $\mathrm{ce}_n(\alpha)$ become the eigenfunctions of the resulting equation:

$$G(\beta)=\int_{-\pi}^{\pi}F(\alpha)Z_0(k|\cos\alpha-\cos\beta|)d\alpha.$$

Series are given which could be used to compute the eigenvalues. The formal solution is now immediate from expansion of $G(\beta)$ in Mathieu functions. However, the author refrains from discussing conditions on G ensuring convergence of the resulting series for F and G. Integral equations from the class discussed arise in the theory of diffraction by a slit, of oscillating airfoils in a compressible fluid, and of ships of minimum wave resistance.

J. V. Wehausen.

Messel, H. On the theory of a nucleon cascade. Communications Dublin Inst. Advanced Studies. Ser. A. no. 7, iii+103 pp. (1951).

Typical of the equations considered in this theory is the integro-differential equation

$$(*)\quad \frac{\partial}{\partial x}G(\epsilon, u, x)+G(\epsilon, u, x)=\int_0^\infty\int_0^\infty G\left(\frac{\epsilon}{\epsilon_1}, u, x\right)G\left(\frac{\epsilon}{\epsilon_2}, u, x\right)w(\epsilon_1, \epsilon_2)d\epsilon_1 d\epsilon_2$$

together with boundary conditions $G(\epsilon, u, x=0)=u$ if $\epsilon<1$ and zero otherwise. Here $G(\epsilon, u, x)$ is the generating function,

$$G(\epsilon, u, x)=\sum_{n=0}^{\infty} u^n\varphi(\epsilon, u, x),$$

for the frequency function $\varphi(\epsilon, u, x)$ governing the probability of finding n nucleons above a given energy ϵ and at a depth x. Also in (*), $w(\epsilon_1, \epsilon_2)$ is a normalized frequency function: $\int_0^\infty\int_0^\infty w(\epsilon_1, \epsilon_2)d\epsilon_1 d\epsilon_2=1$.

From (*), equations for the various n-moments can be derived. Thus differentiating with respect to u and setting $u=1$ we find

$$\frac{\partial}{\partial x}\bar{N}(\epsilon, x)+\bar{N}(\epsilon, x)=\int_0^\infty \bar{N}\left(\frac{\epsilon}{\epsilon_1}, x\right)W(\epsilon_1)d\epsilon_1$$

where

$$W(\epsilon)=\int_0^\infty [w(\epsilon, \epsilon_2)+w(\epsilon_2, \epsilon)]d\epsilon_2=\int_0^\infty \bar{w}(\epsilon, \epsilon_2)d\epsilon_2,$$

and $\bar{N}(\epsilon, x)=\sum n\varphi(\epsilon, n, x)$. Similarly, by differentiating twice with respect to u, we find the equation governing the second moment:

$$\frac{\partial}{\partial x}T(\epsilon, u, x)+T(\epsilon, u, x)=\int_0^\infty T\left(\frac{\epsilon}{\epsilon_1}, x\right)W(\epsilon_1)d\epsilon_1+F(\epsilon, x),$$

where

$$F(\epsilon, x)=\int_0^\infty\int_0^\infty \bar{N}(\epsilon/\epsilon_1, x)\bar{N}(\epsilon/\epsilon_2, x)\bar{w}(\epsilon_1, \epsilon_2)d\epsilon_1 d\epsilon_2,$$

and $T(\epsilon, x)=\bar{N}^2(\epsilon, x)-\bar{N}(\epsilon, x)$. The author shows how equations like the foregoing can be solved by taking their Mellin transforms. The resulting equations are elementary and can be solved explicitly. Taking the inverse Mellin transform of these solutions we obtain the required solutions. The author shows how the complex integrals expressing the inverse Mellin transforms can be evaluated in practical cases by a saddle-point method.

The method which is described above relates to the development of a nucleon cascade in homogeneous matter; in the paper it is extended to include the development of a cascade in non-homogeneous matter. The solutions obtained are illustrated by a number of tables and graphs.

S. Chandrasekhar (Williams Bay, Wis.).

Functional Analysis, Ergodic Theory

***Nachbin, Leopoldo. Some problems of functional analysis.** Symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Diciembre, 1951, pp. 15–21. Centro de Cooperación Científica de la Unesco para América Latina, Montevideo, Uruguay, 1952. (Spanish)

Grothendieck, Alexandre. Sur une notion de produit tensoriel topologique d'espaces vectoriels topologiques, et une classe remarquable d'espaces vectoriels liée à cette notion. C. R. Acad. Sci. Paris **233**, 1556–1558 (1951).

Let E and F be locally convex, linear topological spaces, and let $E\otimes F$ denote the tensor product, that is, the set of all finite linear sums of products. Then $E\otimes F$ can be so topologized that its completion $\widehat{E\otimes F}$ has for dual precisely all numerical-valued $\varphi(x, y)$ which are linear in $x\in E$ and in $y\in F$ and continuous in (x, y). Also $E\otimes F$ can be topologized so that its completion $\overline{E\otimes F}$ has for dual precisely the linear $\varphi(x, y)$ which are continuous in x and in y separately. The two completions need not coincide but there is a natural linear continuous mapping of $\overline{E\otimes F}$ into $\widehat{E\otimes F}$. If E and F are complete, there is a continuous linear mapping of $\widehat{E\otimes F}$ into the set of continuous linear operators from F' (with strongest topology) to E. The operators which correspond in this mapping to the elements in $\overline{E\otimes F}$ are called operators with trace. These notions are analysed further in some special types of E and F including the types introduced by L. Schwartz in his theory of distributions. The author will publish elsewhere an extension of the Fredholm theory to the general case of operators with trace. *I. Halperin.*

Michael, Ernest. Transformations from a linear space with weak topology. Proc. Amer. Math. Soc. **3**, 671–676 (1952).

The main result in this note amounts to the following. Let $\{F_T\}_{T\in K}$ be a family of topological vector spaces F_T and $F=\prod_{T\in K}F_T$ the cartesian product space. Let G be a normed vector space (or, more generally, a topological vector space with some neighborhood of 0 containing no one-dimensional vector subspace). Every linear continuous mapping $F\to G$ depends only on a finite set $\mathfrak{J}\subset K$. Moreover, for any such $\mathfrak{J}$, let $F(\mathfrak{J})$ be the vector subspace of F of all $\{x_T\}_{T\in K}$ such that $x_T=0$ if T non-$\in\mathfrak{J}$. If E is a vector subspace of F, for every linear continuous mapping $E\to G$ to have a linear continuous extension $F\to G$ it is sufficient (but not necessary) that, for every finite $\mathfrak{J}\subset K$, all linear continuous mappings $E\cap F(\mathfrak{J})\to G$ have a linear continuous extension $F(\mathfrak{J})\to G$. This includes Dixmier's remark that a strongly closed vector space of operators in Hilbert space is weakly closed. *L. Nachbin* (São José dos Campos).

Nikodým, Otton M. Criteria for the continuity of linear functionals in real linear spaces. Ohio J. Sci. **52**, 305–313 (1952).

To some extent, the results and methods of this paper are included among those in a paper of the reviewer [Duke Math. J. 18, 443–446 (1951); these Rev. 13, 354]. However, the following result appears to be new: Suppose L is a real linear space with an associated topology which is locally star-like and such that every translate and every non-zero multiple of an open set is open (or, in particular, suppose L is a linear topological space). Then a linear functional on L is continuous if and only if there is a non-empty open set G in L and a real number a such that f does not attain the value a on G. *V. L. Klee* (Charlottesville, Va.).

Köthe, Gottfried. Die Randverteilungen analytischer Funktionen. Math. Z. **57**, 13–33 (1952).

The following notations are employed: Ω is the extended complex plane; C is a simple closed rectifiable Jordan curve, not going through ∞, and usually assumed to be analytic; $\mathfrak{C}$ is the interior of C; $\mathfrak{A}(C)$ is the class of all functions $x(z)$

which are analytic, each on some neighborhood of C; $P(\mathfrak{C})$ and $R(\bar{\mathfrak{C}})$ are respectively the classes of all functions analytic in the open region $\mathfrak{C}$ and on the closed region $\bar{\mathfrak{C}}$; $P(\Omega-\bar{\mathfrak{C}})$, $R(\Omega-\mathfrak{C})$ are the corresponding classes for the exterior regions, with the additional requirement that $x(\infty)=0$ for functions in these last two classes.

$\mathfrak{A}(C)$ is made into a locally convex linear topological space in such a way that convergence in the topology means uniform convergence on some neighborhood of C. The dual space of continuous linear functionals is denoted by $\mathfrak{A}'(C)$. The space $\mathfrak{A}'(C)$ is topologized so that convergence means uniform convergence on the bounded subsets of $\mathfrak{A}(C)$. The spaces $R(\bar{\mathfrak{C}})$ and $R(\Omega-\mathfrak{C})$ are topologized by the same procedure used in topologizing $\mathfrak{A}(C)$. An element of $\mathfrak{A}(C)$ can be decomposed into a sum of an element of $R(\bar{\mathfrak{C}})$ and an element of $R(\Omega-\mathfrak{C})$, by use of Cauchy's integral. This is like the Laurent decomposition of a function analytic on an annulus. The result is that $\mathfrak{A}(C)$ is topologically isomorphic to the direct sum $R(\bar{\mathfrak{C}})\oplus R(\Omega-\mathfrak{C})$. Likewise, with suitable topologies on $P(\mathfrak{C})$ and $P(\Omega-\bar{\mathfrak{C}})$, it turns out that $\mathfrak{A}'(C)$ is topologically isomorphic to $P(\Omega-\bar{\mathfrak{C}})\oplus P(\mathfrak{C})$. The component isomorphisms are

$$R'(\bar{\mathfrak{C}})\cong P(\Omega-\bar{\mathfrak{C}}),\quad R'(\Omega-\mathfrak{C})\cong P(\mathfrak{C}).$$

These results are found by using the Fantappiè indicatrix of a functional.

Results parallel to the foregoing are obtained for the spaces $\mathfrak{E}(C)$ and its dual $\mathfrak{E}'(C)$, where C is assumed to be analytic and $\mathfrak{E}(C)$ is the class of all functions defined and having derivatives of all orders on C, with respect to the parameter along C. With the topology assigned to $\mathfrak{E}(C)$, the dual space consists of distributions in the sense of L. Schwartz. As a class, $\mathfrak{A}(C)$ is dense in $\mathfrak{E}(C)$, and $\mathfrak{E}'(C)$ is a subset of $\mathfrak{A}'(C)$.

The author has some things to say regarding elements of $\mathfrak{A}'(C)$ as "boundary distributions" on C. The corresponding analytic functions need not actually approach boundary values on C in any point-by-point sense, however.

A. E. Taylor (Los Angeles, Calif.).

Lorentz, G. G. Multiply subadditive functions. Canadian J. Math. 4, 455–462 (1952).

The fact that a functional p on a real vector space X is a pseudo-norm if and only if it has the form

$$p(x)=\sup_f\{|f(x)|\},$$

where f ranges over a fixed non-empty class of linear functionals, is translated here to the theorem that a functional p on a Boolean ring R has a property called multiple subadditivity if and only if it has the form $p(e)=\sup_f\{|f(e)|\}$, where f ranges over a fixed non-empty class C of additive functionals (measures) on R. A nonnegative function p is said to be multiply subadditive if and only if $np(e)\leq\sum p(e_i)$ whenever the finite set $\{e_i\}$ covers e at least n times; this is the condition that p be extendable to a norm on the vector lattice of "step-functions on R". The resolution of an additive functional f into its variations allows $|f(e)|$ to be replaced by $f(e)$ in the above theorem, provided C is now restricted to positive functionals. The author also shows that if R is the algebra of measurable subsets e of $[0, 1]$ and m is Lebesgue measure, then p is of the form $p(e)=F(m(e))$, where F is continuous, increasing, concave, and zero at the origin, if and only if the family C can be taken to be the family of all rearrangements (under measure-preserving transformations of $[0, 1]$) of a fixed absolutely continuous set function.

L. H. Loomis (Cambridge, Mass.).

Halperin, Israel. The supremum of a family of additive functions. Canadian J. Math. 4, 463–479 (1952).

The author generalizes the main theorem of Lorentz [see the paper reviewed above] by replacing the Boolean ring R by a general commutative semi-group G. Instead of imbedding G in a real vector space and using the Hahn-Banach theorem there, the author proceeds by proving the needed generalization of this theorem directly for G. The semi-group G is then generalized to a system S having a partially defined addition operation. It is shown that if S is a relatively complemented modular lattice with zero, in which perspectivity is transitive, then the theorem can be stated in terms of a concept more nearly like that of "being covered n times" used by Lorentz. The exact conditions of the paper are too complicated to be given here.

L. H. Loomis (Cambridge, Mass.).

Schoenberg, I. J. A remark on M. M. Day's characterization of inner-product spaces and a conjecture of L. M. Blumenthal. Proc. Amer. Math. Soc. 3, 961–964 (1952).

A vector space S is said to be "semi-normed" provided a real function $\|f\|$ is defined on S satisfying $\|o\|=0$, $\|f\|=\|-f\|>0$ when $f\neq o$. The semi-norm $\|f\|$ is "Ptolemaic" if for any four vectors a, b, c, d Ptolemy's inequality

$$\|b-a\|\cdot\|d-c\|+\|d-a\|\cdot\|c-b\|\geq\|c-a\|\cdot\|d-b\|$$

always holds. Theorem. Any Ptolemaic semi-normed space is a real inner-product space. Sketch of the proof. Ptolemy's inequality is applied to $a=o$, $b=f$, $c=(f+g)/2$, $d=g$ to prove the norm inequality $\|f+g\|\leq\|f\|+\|g\|$, and to $a=f$, $b=g$, $c=-f$, $d=-g$ to prove $\|f-g\|^2+\|f+g\|^2\geq4\|f\|\cdot\|g\|$, hence $\|f-g\|^2+\|f+g\|^2\geq4$ if $\|f\|=\|g\|=1$. An adaptation of Day's geometrical procedure [Trans. Amer. Math. Soc. 62, 320–337 (1947); these Rev. 9, 192] yields his condition $\|f-g\|^2+\|f+g\|^2=4$ on the unit sphere.

Chr. Pauc.

Miyadera, Isao. On one-parameter semi-group of operators. J. Math. Tokyo 1, 23–26 (1951).

The author gives an independent proof of the fact that if $T(\xi)$, $\xi>0$, is a strongly measurable semi-group of linear operators on a (B)-space to itself, then $\|T(\xi)\|$ is bounded in each interval $[\delta, 1/\delta]$. This result has also been obtained by the reviewer [Proc. Amer. Math. Soc. 2, 234–237 (1951); these Rev. 12, 617].

R. S. Phillips (Los Angeles, Calif.).

Miyadera, Isao. Generation of a strongly continuous semi-group operators. Tôhoku Math. J. (2) 4, 109–114 (1952).

Let $T(\xi)$, $0\leq\xi<\infty$, be a one-parameter semi-group of linear bounded transformations on a (B)-space $\mathfrak{X}$ to itself satisfying the conditions: $T(\xi+\eta)=T(\xi)T(\eta)$, $T(0)=I$, and $T(\xi)x\to x$ as $\xi\to0$ for all $x\in\mathfrak{X}$. Denote the infinitesimal generator of $[T(\xi)]$ by A and the resolvent of A by $R(\lambda; A)$. The author shows that a necessary and sufficient condition for a closed linear operator A with dense domain to generate a semi-group of the above type is that there exist real numbers $M>0$ and ω such that $\|[R(\lambda; A)]^n\|\leq M(\lambda-\omega)^{-n}$ for $\lambda>\omega$ and $n>0$. The method of proof follows K. Yosida's proof for the case $M=1$ [J. Math. Soc. Japan 1, 15–21 (1948); these Rev. 10, 462].

R. S. Phillips.

Sz.-Nagy, Béla. On the stability of the index of unbounded linear transformations. Acta Math. Acad. Sci. Hungar. 3, 49–52 (1952). (Russian summary)

Let E, E' be Banach spaces with norms $\|\ \|$, $\|\ \|'$ and let A be a closed linear operator on E with closed range in E' for

which the index $\nu(A)$ can be defined. It is proved (1) that there is a positive ρ such that if the linear operator B satisfies $\|Bf\|'\leq\rho(\|f\|+\|Af\|')$, then $\nu(A+B)=\nu(A)$, and (2) that if B is completely continuous, then $\nu(A+B)=\nu(A)$. This improves on results proved by M. G. Kreĭn and M. A. Krasnosel'skiĭ [Mat. Sbornik N.S. **30**(72), 219–224 (1952); these Rev. **13**, 849] in extension of results of the reviewer [ibid. **28**(70), 3–14 (1951); these Rev. **13**, 46]. It is shown that the renorming of E given by $|f|=\|f\|+\|Af\|'$ reduces the problems to the more special case considered by the reviewer. *F. V. Atkinson* (Ibadan).

★**Sobolev, S. L. Nekotorye primeneniya funkcional'nogo analiza v matematičeskoĭ fizike.** [**Some applications of functional analysis in mathematical physics.**] Izdat. Leningrad. Gos. Univ., Leningrad, 1950. 255 pp. 16 rubles.

Chap. I, Special questions of functional analysis: Introduction; Fundamental properties of spaces L_p; Linear functionals in L_p; Compactness of spaces; Generalized derivatives; Properties of integrals of the type of a potential; Spaces $L_p^{(l)}$ and $W_p^{(l)}$; Embedding theorems; General methods of norming $W_p^{(l)}$ and consequences of an embedding theorem; Some consequences of embedding theorems; Complete continuity of the embedding operator (theorem of Kondrašev). Chap. II, Variational methods in mathematical physics: Dirichlet's problem; Neumann's problem; Polyharmonic equation; Uniqueness of solution of a fundamental boundary problem for the polyharmonic equation; Problem of characteristic values. Chap. III, Theory of hyperbolic differential equations: Solution of the wave equation with smooth initial conditions; Generalized Cauchy problem for the wave equation; Linear equation of normal hyperbolic type with variable coefficients (basic properties); Cauchy's problem for linear equations with smooth coefficients; Investigation of linear hyperbolic equations with variable coefficients; Quasi-linear equations. *Table of contents.*

★**Halmos, Paul R. Some present problems on operators in Hilbert space.** Symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Diciembre, 1951, pp. 9–14. Centro de Cooperación Científica de la Unesco para América Latina, Montevideo, Uruguay, 1952. (Spanish)

Katětov, Miroslav. Linear operators. II. Časopis Pěst. Mat. **76**, 105–119 (1951). (Czech)

The author continues his expository article [same Časopis **75**, D9–D31 (1950); these Rev. **12**, 419] with paragraphs on bilinear forms, especially hermitean forms, adjoint and normal operators, and projections. *František Wolf.*

Stone, M. H. On unbounded operators in Hilbert space. J. Indian Math. Soc. (N.S.) **15** (1951), 155–192 (1952).

The graph $\mathfrak{G}=\mathfrak{G}(A)$ of an arbitrary operator A in Hilbert space $\mathfrak{H}$ was defined in 1932 by J. von Neumann as the set of all vectors $\{x_1, x_2\}$ in the space $\mathfrak{H}\oplus\mathfrak{H}$ such that $x_2=Ax_1$; this concept proved to be very successful, especially in the study of the adjoint operators. Each bounded linear operator S in $\mathfrak{H}\oplus\mathfrak{H}$ is uniquely expressible through the relation

$$S\{x_1, x_2\}=\{S_{11}x_1+S_{12}x_2, S_{21}x_1+S_{22}x_2\},$$

in terms of a two-rowed matrix (S_{ik}) of bounded linear operators in $\mathfrak{H}$. The matrix (P_{ik}) of the projection P of $\mathfrak{H}\oplus\mathfrak{H}$ onto the subspace generated by $\mathfrak{G}(A)$ is called the characteristic matrix of A. If P is a projection of $\mathfrak{H}\oplus\mathfrak{H}$ onto a subspace $\mathfrak{M}$, then $\mathfrak{M}$ is the graph of an operator A if and only if $P_{12}x=0$ and $P_{22}x=x$ imply $x=0$. Then A is necessarily a (uniquely determined) closed linear operator, its domain consists of all vectors of the form $x=P_{11}x_1+P_{12}x_2$ where x_1, x_2 are arbitrary vectors of $\mathfrak{H}$, and we have then $Ax=P_{21}x_1+P_{22}x_2$. In particular, the identities $P_{21}=AP_{11}$, $P_{22}=AP_{12}$ hold.

The first section of the paper gives, by a systematic use of the characteristic matrices, a somewhat more elementary derivation of the principal properties of nonbounded linear operators, first established by von Neumann, and also gives some criteria, in terms of the characteristic matrix, in order that the operator A be bounded, or self-adjoint, or normal, or unitary, etc.

The following section deals with the commutants $\mathbf{A}'$ of any set $\mathbf{A}$ of operators; $\mathbf{A}'$ consists of all those bounded linear operators B which, together with their adjoints, commute with every operator $A\in\mathbf{A}$, i.e., $BA\subset AB$, $B^*A\subset AB^*$. The elementary theory of commutants, developed originally by von Neumann, is briefly reviewed and extended to the case where $\mathbf{A}$ consists of nonlinear, or nonclosed, operators, and also to the case where the space $\mathfrak{H}$ admits quaternions as scalars. In the proofs, essential use is made of the characteristic matrices.

The final section is devoted to a discussion of commutativity for nonbounded operators. The fundamental definition made is the following. A set $\mathbf{A}$ of operators in $\mathfrak{H}$ is said to be abelian if $\mathbf{A}'\supset\mathbf{A}'''\supset\mathbf{A}''$, or, equivalently, if $\mathbf{A}''$ is a commutative algebra (of bounded linear operators) and $\mathbf{A}'=\mathbf{A}'''$. An operator A is said to be abelian if the set (A) is abelian. It is proved that: (i) when $\mathbf{A}$ consists of closed linear operators, it is abelian if and only if $\mathbf{A}''$ is commutative; (ii) when $\mathbf{A}$ consists of bounded linear operators, it is abelian if and only if the operators in $\mathbf{A}$, together with their adjoints, commute with one another; (iii) a closed linear operator A is abelian if and only if it is normal in a subspace of $\mathfrak{H}$; (iv) a set $\mathbf{A}$ of closed linear operators is abelian if and only if each of them is normal in a subspace of $\mathfrak{H}$ and the components of the characteristic matrices commute.

Finally, without any appeal to the spectral theorem, the following commutation criterion for an operator to have a normal extension is proved: Let $\mathfrak{H}$ be a real or complex Hilbert space, $\mathbf{C}$ any abelian set, and $\mathbf{A}$ any set of operators such that $\mathbf{A}'\supset\mathbf{C}'$. Then each $A\in\mathbf{A}$ has in the subspace generated by its domain $\mathfrak{D}_A$ a unique normal extension $\tilde{A}$. If $\mathfrak{D}_A$ generates $\mathfrak{H}$, then $\tilde{A}$ is represented in the form $B^{-1}C$ with B, $C\in\mathbf{C}''$, B being an invertible self-adjoint operator, and C being a normal operator which commutes with B and satisfies the identity $CC^*=B-B^2$. This theorem is called the S.O.M. theorem because it represents a generalization of the following theorem, due to U. Sasaki and J. Ogasawara [J. Sci. Hiroshima Univ. Ser. A. **6**, 271–278 (1936)] and to Y. Mimura [Jap. J. Math. **13**, 119–128 (1937)]: Let H be a self-adjoint operator in the complex Hilbert space $\mathfrak{H}$, and let A be an operator such that $(A')\supset(H)'$. Then A has a normal extension which is, in the sense of the calculus of operators, a function of H. *B. Sz.-Nagy* (Szeged).

Lifšic, I. M. On regular perturbations of an operator with a quasi-continuous spectrum. Učenye Zapiski Har'kov. Gos. Univ. **28**, Zapiski Naučno-Issled. Inst. Mat. Meh. i Har'kov. Mat. Obšč. (4) **20**, 77–82 (1950). (Russian)

The author considers a sequence of hermitian operators which have a discrete spectrum and converge in a certain way to an operator with continuous spectrum. He uses formulae deduced in a previous paper [C. R. (Doklady)

Acad. Sci. URSS (N.S.) 48, 79–81 (1945); these Rev. 7, 453] for the perturbation of a continuous spectrum. His "quasi-continuous" spectrum refers really to the whole sequence which serves for this particular purpose. The paper is devoid of details and only sketches results obtained by doubtful limiting processes. The most important seem to be the emergence of isolated spectral singularities of the perturbed operator which detach themselves from the end-points of the continuous spectrum. These results seem similar to those obtained previously by K. O. Friedrichs [Communications on Appl. Math. 1, 361–406 (1948); these Rev. 10, 547]. *František Wolf* (Berkeley, Calif.).

Misonou, Yosinao. On a weakly central operator algebra. Tôhoku Math. J. (2) 4, 194–202 (1952).

All algebras considered have complex scalars and an identity element. Let A be a C^*-algebra (a uniformly closed self-adjoint algebra of operators on a Hilbert space) which is weakly central (i.e., M_1 and M_2 are maximal ideals and $M_1\cap Z=M_2\cap Z$ imply $M_1=M_2$, where Z is the center of A). Let Ω be the spectrum of A (the set of maximal ideals in A with the Stone topology). Then Ω is homeomorphic with the spectrum of Z under the mapping $M\to M\cap Z$. Therefore Ω is a compact Hausdorff space. For ζ a maximal ideal in Z, let I_ζ be the intersection of all closed ideals of A containing ζ. Let $A_\zeta=A/I_\zeta$. Let B be the algebra of functions f on Ω such that $f(\zeta)\in A_\zeta$ for each $\zeta\in\Omega$. The author proves that there is a $*$-isomorphism $x\to x(\zeta)$ of A into B such that (1) $\|x\|=\sup\|x(\zeta)\|$, (2) $\|x(\zeta)\|$ is continuous on Ω for each $x\in A$, and (3) if C is a subalgebra of B which contains the image of A and $\|\theta(\zeta)\|$ is continuous on Ω for each $\theta\in C$, then C is the image of A. In the proof of (3) the author uses only "continuity at 0". The author shows that every W^*-algebra (weakly closed self-adjoint algebra of operators on a Hilbert space) is weakly central. Two results on the existence of projections in W^*-algebras are proved as applications. *J. A. Schatz* (Bethlehem, Pa.).

Hausner, M., and Wendel, J. G. Ordered vector spaces. Proc. Amer. Math. Soc. 3, 977–982 (1952).

The authors seek to represent an arbitrary ordered vector space in some canonical form. They attribute to R. Thrall the following definition of a lexicographic function space: Let T be any ordered set of indices, let f range over all functions which are real-valued for each index in T and which take nonzero values on at most a well-ordered subset of T. Let V_T be the linear space of such functions ordered according to the relation: $f>g$ if $f(t)>g(t)$ at the first index t in T for which inequality occurs. Examples are given to show that not every ordered vector space is isomorphic with some lexicographic function space, but the main result is that every ordered vector space is isomorphic to a subset V_T' of some V_T and this subset can be specialized to have the following properties: (i) V_T' contains all f which are nonzero on a single index of T, and (ii) whenever f is in V_T' and t_0 is an arbitrary index in T, then the truncated function $g(t)$ which coincides with $f(t)$ for $t<t_0$ and vanishes for $t\geqq t_0$ is also in V_T'. *I. Halperin* (Kingston, Ont.).

***Cotlar, Mischa. On the foundations of ergodic theory.** Symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Diciembre, 1951, pp. 71–84. Centro de Cooperación Científica de la Unesco para América Latina, Montevideo, Uruguay, 1952. (Spanish)

Expository account of certain aspects of ergodic theory.

Theory of Probability

***Boev, G. P. Teoriya veroyatnosteĭ. [The theory of probability.]** Gosudarstv. Izdat. Tehn.-Teor. Lit. Moscow-Leningrad, 1950. 368 pp. 9.45 rubles.

This book is a textbook, based on a course given by the author at Saratov State University, and is accessible to students familiar with the calculus. The book is written in the classical tradition, in that probability is not presented as a purely mathematical subject with an empirical interpretation, but rather as a systematic analysis of certain types of empirical phenomena. The chapters are: 1) Probability. 2) Random variables. 3) Limit theorems. (This chapter includes proofs of the weak law of large numbers and of the central limit theorem.) 4) Probability of hypotheses. (Much of the discussion is based on the convention, whose significance is not discussed, that any parameter to be estimated has a uniform a priori distribution.) 5) Stochastic processes. (In this chapter an inexperienced reader would find it difficult to decide which parts of the discussion refer only to Markov processes.) There is a concluding section with a few remarks on the further development of the subject.

The book is too short to give more than an introduction to each topic treated, but the book is as rigorous as can be reasonably expected at this level. The discussion of statistical concepts is decidedly old-fashioned. *J. L. Doob.*

Sibirani, Filippo. Sopra un problema di probabilità. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) 7 (1949–50), 143–146 (1951).

For k drawings from an urn containing balls numbered 1 to N, the probability of obtaining a set of balls such that the ball drawn jth is divisible by a_j is shown to be the same with and without replacement for $k=2$, 3 if and only if $N\equiv 0 \bmod (a_1a_2\cdots a_k)$ and a_1 to a_k are relatively prime; and the same thing is conjectured in general.

J. Riordan (New York, N. Y.).

de Castro, Gustavo. Note on differences of Bernoulli and Poisson variables. Portugaliae Math. 11, 173–175 (1952).

The distribution of the difference of two Bernoulli (Poisson) variates is expressed in terms of Legendre (Bessel) functions.

E. Lukacs (Washington, D. C.).

Kanellos, S. G. On a conditional distribution. Bull. Soc. Math. Grèce 26, 24–28 (1952). (Greek summary)

Let E be an event of probability p. Let c be the number of occurrences of E in $2k$ trials $e_1, \cdots, e_{2k}$ and let s be the number of occurrences of the joint event (E, E) in the k paired trials $(e_1, e_2), \cdots, (e_{2k-1}, e_{2k})$. If the trials e_i are independent of one another, then the conditional probability distribution for s knowing c depends on k and c but not on p. Hence this distribution may be used to test independence of a sequence of trials $e_1, e_2, \cdots$. The following expression is derived for the rth factorial moment of the distribution:

$$\mu_r=\binom{2k-2r}{c-2r}k!\Big/\binom{2k}{c}(k-r)!.$$

E. N. Gilbert (Murray Hill, N. J.).

Marczewski, E. Un théorème de S. Mazurkiewicz sur les espaces de variables aléatoires. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. **41** (1948), 7–9 (1950). (Polish. French summary)

The author states that an unpublished result of S. Mazurkiewicz concerning the representation of a separable family of random variables as a family of Lebesgue measurable functions defined on the unit interval can be obtained by means of a theorem in measure theory to the effect that the Lebesgue measure on the unit interval is in a certain sense universal for all separable measures. This result of S. Mazurkiewicz has since been published (completed and commented on by E. Marczewski) in Fund. Math. **36**, 288–302 (1949) [these Rev. **12**, 108]. *S. Kakutani.*

Huron, R. Sur la répartition des décimales de rang donné dans les tables numériques. Ann. Fac. Sci. Univ. Toulouse (4) **15**, 161–186 (1951).

Let $f(x)$ be a continuous monotone positive function in the interval (a, b). Divide (a, b) by the arithmetic means $a_r = a + ((b-a)/n)r$, $r = 0, 1, \cdots, n$. The author studies the distribution of the ith decimal digit of the values $f(a_r)$, $r = 0, 1, \cdots, n$. A method for obtaining the limiting distribution as $n \to \infty$, i fixed, is given. Special cases are discussed. In some cases the mean only is given and in others the actual limiting distribution is obtained. If $\lim_{n\to\infty} f(n)/n = 0$, the method applies to the problem of finding the limiting distribution of the ith decimal digit in the values of $f(n)$ for integers n less than x, the limit taken as $x \to \infty$. In this case the logarithm does not give a uniform distribution but the square root does. *J. L. Snell* (Princeton, N. J.).

Lur'e, A. L. On an inverse Bernoulli theorem. Doklady Akad. Nauk SSSR (N.S.) **50**, 45–48 (1945). (Russian)

The probability of occurrence of an event in a sequence of independent trials is a random variable p with an a priori distribution $F(x)$. The number of occurrences in the first trials is μ_n. It is shown that the limit of the inverse probability $P(|n^{-1}m_n - p| < \epsilon | \mu_n = m_n)$ tends to 1 as $n \to \infty$, uniformly with respect to every sequence m_n, if and only if $F(x)$ is strictly increasing in $(0, 1)$. *K. L. Chung.*

Sakaguchi, Minoru. On a certain limit distribution. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. **1**, no. 4, 10–14 (1952).

Ugaheri's problem [Ann. Inst. Statist. Math., Tokyo **1**, 157–160 (1950); these Rev. **11**, 731] is generalized to an initial density which is symmetrical in $(0, 1)$. *K. L. Chung* (Ithaca, N. Y.).

Dynkin, E. B. Criteria of continuity and of absence of discontinuities of the second kind for trajectories of a Markov random process. Izvestiya Akad. Nauk. SSSR. Ser. Mat. **16**, 563–572 (1952). (Russian)

Let x_t be a Markov process with the transition function $P(s, x, t, E)$ where the state space Ω is complete in the metric ρ and t ranges over a closed interval I. Let $V_\epsilon(x)$ be the set of y such that $\rho(x, y) \geqq \epsilon$; $\varphi_\epsilon(h) = \sup P(t, x, t+\alpha, V_\epsilon(x))$ where the sup is taken over all $t, t+\alpha \in I$, $x \in \Omega$, and $0 \leqq \alpha \leqq h$. Theorem 1: If $\varphi_\epsilon(h) = o(h)$ for every $\epsilon > 0$, then x_t is continuous in I with probability one. Theorem 2: If $\varphi_\epsilon(h) = o(h^{1/2})$ for every $\epsilon > 0$, then x_t has no discontinuity of the second kind, with probability one. The proof depends on elementary estimates and is carried through first for a denumerable set of t in I, then completed by Doob's method of extending the probability measure. Alternatively, according to Doob's more recent approach, one may take the process to be separable to begin with. Lévy's theorem for processes with independent increments and Wiener's theorem for the Brownian movement process are consequences. Easy extensions are mentioned. *K. L. Chung* (Ithaca, N. Y.).

Dobrušin, R. L. On conditions of regularity of stationary Markov processes with a denumerable number of possible states. Uspehi Matem. Nauk (N.S.) **7**, no. 6(52), 185–191 (1952). (Russian)

Let $[P_i^j(t)]$ be the transition probability matrix function of a stationary Markov chain. It is supposed that $P_i^{i\prime}(0) = -a_i$, $P_i^{j\prime}(0) = a_i^j = a_i\pi_i^j$ $(j \neq i)$ exist and are finite, with $\sum_{j\neq i} a_i^j = a_i$. Set $\pi_i^i = 0$. A simple proof due to Dynkin is given of Feller's necessary and sufficient condition [Trans. Amer. Math. Soc. **3**, 488–515 (1940); these Rev. **2**, 101] that (R) almost all sample functions of the process have at worst jumps as discontinuities, for every initial state. If $x_1, x_2, \cdots$ are the successive states of a Markov discrete parameter chain with the transition matrix $[\pi_i^j]$, it is proved that the given process satisfies (R) if and only if $\sum_n 1/a_{x_n} = \infty$ with probability 1 for every initial state. If $\{a_i\}$ is given, a chain determined by $\{a_i\}$ and $[\pi_i^j]$ satisfies (R) for every choice of $[\pi_i^j]$ if and only if the sequence $\{a_i\}$ is bounded. If $[\pi_i^j]$ is given, the chain satisfies (R) for every choice of $\{a_i\}$ if and only if every state of the chain determined by $[\pi_i^j]$ has positive probability of going into some recurrent state of this chain. (According to the author, this theorem was proved simultaneously by Vedenskiĭ.) Finally, if $a_i^j = 0$ for $j \geqq i+2$ and if $a_i^{i+1} > 0$, the process satisfies (R) if and only if, when the initial state is 1, the total time to the first sample function discontinuity which is not a jump has expectation $+\infty$. *J. L. Doob* (Urbana, Ill.).

Harris, T. E. First passage and recurrence distributions. Trans. Amer. Math. Soc. **73**, 471–486 (1952).

Soit une chaîne de Markoff homogène discrète à une infinité dénombrable d'états possibles $E_0, E_1, \cdots, E_i, \cdots$; on pose: X_n = la variable aléatoire qui vaut i si à l'instant n l'état réalisé est E_i; $P^n{}_{ij} = \Pr(X_{n+h} = j | X_n = i)$; si $X_n = i$, $N^n{}_{ij}$ = le plus petit entier positif R tel que $X_{n+h} = j$; $\theta_{ij} = \Pr(\text{qu'il existe un } h < N^n{}_{ii} \text{ tel que } X_{n+h} = j | X_n = i)$ $(i \neq j)$. On suppose que pour chaque couple (i, j) il existe un entier n tel que $P^n{}_{ij} > 0$ et que $E(N_{ii}) < +\infty$, ce qui entraîne l'existence de $\pi_j > 0$ $(j = 0, 1, 2, \cdots)$ tels que:

$$\pi_j = \lim_{n\to+\infty} n^{-1}\sum_{h=1}^{n} P^h{}_{ij} = \sum_{k=0}^{+\infty} \pi_k P^1{}_{kj}, \quad \sum_{k=0}^{+\infty} \pi_k = 1.$$

L'auteur démontre que

$$\lim_{k\to+\infty} \Pr(\pi_i\theta_{ik}N^n{}_{ii} > u) = e^{-u}, \quad \lim_{k\to+\infty} \Pr(\pi_k\theta_{ki}N^n{}_{kk} > u) \sim \theta_{ki}e^{-u}$$

$(i, n$ fixes quelconques, $u > 0$ fixe quelconque). Soit en particulier: $P^1{}_{i,i+1} = p_i$, $P^1{}_{i,i-1} = 1 - p_i$ $(i > 0)$ ("promenade au hasard"); on pose: $L_1 = 1$, $L_{j+1} = \prod_{i=1}^{j}(1-p_i)/p_i$, $z_0 = 0$, $z_j = \sum_{i=1}^{j} L_i$, $Z = \sum_{i=1}^{+\infty} z_i$, $U = \sum_{i=1}^{+\infty} L_i \leqq +\infty$; l'auteur démontre que $\Pr(N^n{}_{i0} < +\infty) = z_i/U$, si $0 < p_i < 1$ pour $i = 1, 2, \cdots$, et que $\Pr(N^n{}_{i\omega} < N^n{}_{i0}) = z_i/z_\omega$ si $0 < p_j < 1$ pour $j = 1, 2, \cdots, \omega - 1$, si $p_\omega = 0$ et si $i \leqq \omega$; il obtient quelques autres résultats, en particulier une équation de recurrence pour la fonction génératrice de $\frac{1}{2}(1 + N_{i,i+1})$, et montre l'identité du processus considéré de promenade au hasard et de certains processus ramifiés. *R. Fortet* (Paris).

Vallée, Robert. Sur deux classes d' "opérateurs d'observation." C. R. Acad. Sci. Paris 233, 1350–1351 (1951).

By "observation operator" is meant an operator which transforms a physical quantity $f(x, y, z, t)$ into another function $g(x, y, z, t)$ representing the result of an experiment to measure f. A brief description is given of two kinds of observation operators which are asserted to have an important connection with the theory of information.
E. N. Gilbert (Murray Hill, N. J.).

Itô, Hirosi. On the theory of continuous information. Proc. Japan Acad. 28, 187–191 (1952).

The author is disturbed by the fact that the entropy function $H(x)$ of C. E. Shannon [Bell System Tech. J. 27, 379–423, 623–656 (1948); these Rev. 10, 133] may have a negative sign when x is a random variable with a continuous distribution. *E. N. Gilbert* (Murray Hill, N. J.).

Reich, Edgar. On the definition of information. J. Math. Physics 30, 156–161 (1951).

Information symbols are represented by real numbers and $p(x, y)$ is the probability density that x is transmitted and y is received. Let $p_0(x)=\int p(x, y)dy$, $q(y)=\int p(x, y)dx$ and $p_m(x)=p(x, m)/q(m)$. A scalar functional $I(m)$ of $p_0(x)$ and $p_m(x)$ is desired which will have the following two properties: (a) $I(m)=0$ if $p_0(x)$ and $p_m(x)$ are the same; (b) the $E\{I\}=\int g(m)I(m)dm$ remains the same if the densities $p_j(x)$, $j=1, m$, are transformed to $p_j(g(x))/f'(g(x))$ where f is a monotone differentiable function with inverse $g(x)$. This corresponds to a one-to-one relabeling of the message space. The author proves that if I is of the form U_0-U_m, and $U_i=\int F(p_i(x), x)dx$ where F is a sufficiently regular function, then (a) and (b) imply that $E\{I\}$ has the form given by Shannon for information rate, namely,

$$E\{I\}=\text{constant}\times\int\int P(x, y)\log[p(x, y)/p_0(x)q(y)]dxdm.$$

J. L. Snell (Princeton, N. J.).

Fromageot, Antoine. Notion d'entropie en calcul des probabilités. Ann. Télécommun. 7, 388–396 (1952).

An exposition of some parts of C. E. Shannon's paper, Bell System Tech. J. 27, 379–423, 623–656 (1948) [these Rev. 10, 133]. *E. N. Gilbert* (Murray Hill, N. J.).

Kullback, S. A note on information theory. J. Appl. Phys. 24, 106–107 (1953).

Maignan, Paul, Blanc, D., et Detoeuf, J.-F. Théorie élémentaire des fonctions génératrices. Application aux fluctuations statistiques des compteurs à scintillations. J. Phys. Radium (8) 13, 661–667 (1952).

La théorie des fonctions génératrices est succinctement exposée; son application aux fluctuations statistiques d'une installation de couplage par scintillation a été développée dans deux cas: celui des scintillateurs minces et celui des scintillateurs épais. *Authors' summary.*

Pompili, Eraldo. Sul minimo della probabilità di fallimento nelle imprese di assicurazione. Statistica, Bologna 12, 377–382 (1952).

The paper deals (under various restrictive assumptions) with the problem of determining the quota of reinsurance which minimizes the probability of ruin. *E. Lukacs.*

Mathematical Statistics

***Rios, Sixto. Introducción a los metodos de la estadística. 1ª parte.** [Introduction to the methods of statistics. Part 1.] 2d ed. Madrid, 1952. xii+192 pp. 80 pesetas.

Introduction. Statistical tables and graphical representations. Frequency and probability. Sample and universe. Reduction of statistical data. Some distributions. Multidimensional statistical variables. Two-dimensional random variables. Introduction to sampling theory. Sampling distributions of some statistics. Intuitive introduction of tests and comparisons of statistical hypotheses.
Table of contents.

***Quenouille, M. H. Associated measurements.** Academic Press Inc., New York; Butterworths Scientific Publications, London, 1952. x+242 pp. $5.80.

A treatise on correlation analysis. A manual rather than a text-book, the emphasis being on the graphic and numerical illustration of standard procedures, whereas the rationale of these procedures is discussed only briefly. Four sections, each with three chapters: (1) Graphical analysis, (2) Numerical analysis, (3) Rapid estimation and analysis, (4) Analytical complications. The bibliography gives some 130 selected references. The appendix tables cover 13 significance tests, making an unconventional selection of classic and recent material.

The brief treatment of the underlying theories is a drawback, especially in the last two chapters, which in 38 pages deal with as large and diverse topics as time-series analysis, factor analysis, and discriminatory tests. For example, the fundamental distinction between evolutive and stationary time-series is not made, although the exposition includes trend-fitting procedures, which have no place in stationary series, as well as correlogram methods, which have no place in evolutive series. It puzzles the reviewer that the correction factor $f=(1+2r_1r_1'+2r_2r_2'+\cdots)^{1/2}$ is used as a panacea for reducing the degrees of freedom in tests referring to two time-series with serial coefficients r_k and r_k', factor f being recommended for (a) correlation coefficients, (b) product moments, (c) regression coefficients, (d) canonical correlations. In case (a) the device is correct only when testing zero intercorrelation [see E. Slutsky, C. R. Acad. Sci. Paris 189, 612–614 (1929)]. In case (d) its rationale is not clear to the reviewer. The author attributes f to Bartlett, who dealt with a special case of (b) [J. Roy. Statist. Soc. (N.S.) 98, 536–543 (1935); see also Moran, Biometrika 34, 281–291 (1947), p. 285; these Rev. 9, 361, 735]. In case (c) factor f was established by the reviewer [Bull. Inst. Internat. Statist. 32, no. 2, 277–289 (1950); these Rev. 13, 261]. The author communicates that a square root sign is missing before the denominator $2(1-r_{xy})$ on p. 70. *H. Wold.*

Fortunati, Paolo. Appunti sulle misure statistiche della variabilità. Statistica, Bologna 12, 297–321 (1952).

This is a discussion of various measures of variability from the viewpoint of descriptive statistics. *E. Lukacs.*

Skory, John. Automatic machine method of calculating contingency X^2. Biometrics 8, 380–382 (1952).

David, H. A. Upper 5 and 1% points of the maximum F-ratio. Biometrika 39, 422–424 (1952).

Let $\chi_1^2, \chi_2^2, \cdots, \chi_k^2$ be independent chance variables, each having a Chi-square distribution with ν degrees of freedom.

Let F_{max} be the ratio of the largest to the smallest of these χ_i^2. Tables are computed of the upper 5% and 1% points of F_{max}. *H. Chernoff* (Stanford, Calif.).

Homma, Tsuruchiyo. On the limit distributions of some ranges. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. 1, no. 4, 15–26 (1952).

Possible limit laws for the range and the midrange are obtained for a continuous population distribution. It is also shown that the range and the midrange are not asymptotically independent. *K. L. Chung* (Ithaca, N. Y.).

Draper, J. Properties of distributions resulting from certain simple transformations of the normal distribution. Biometrika 39, 290–301 (1952).

This paper is chiefly concerned with the means of effecting the transformation $z=\gamma+\delta \sinh^{-1}((x-\xi)/\lambda)$ of the random variable x to the unit normal variable z in which the parameters γ, δ, ξ, and λ are estimated from a sample. This is the second of a set of transformations studied by N. L. Johnson [Biometrika 36, 149–176 (1949); these Rev. 11, 527] who gave a graphical method for estimation of the parameters. In the present paper approximate algebraic methods are developed, their ranges of applicability are discussed and two numerical illustrations are given. Some possibilities of useful application of Johnson's transformations to sampling distributions are more briefly discussed. In the case of Johnson's third transform (the first was the log-normal), $z=\gamma+\delta \log(y/(1-y))$, where $y=(x-\xi)/\lambda$, the author develops a method for calculating moments in terms of the parameters, using a quadrature formula due to E. T. Goodwin [Proc. Cambridge Philos. Soc. 45, 241–245 (1949); these Rev. 10, 575]. *C. C. Craig* (Ann Arbor, Mich.).

Nabeya, Seiji. Absolute moments in 3-dimensional normal distribution. Ann. Inst. Statist. Math., Tokyo 4, 15–30 (1952).

Given that x_1, x_2, x_3 obey a three-dimension normal distribution law with zero mean, the author gives his method of computing $E(|x_1^{n_1}x_2^{n_2}x_3^{n_3}|)$ for positive integral values of n_1, n_2, n_3 and records his results for all $n_1+n_2+n_3 \leqq 12$.
C. C. Craig (Ann Arbor, Mich.).

Des Raj. On estimating the parameters of normal populations from singly truncated samples. Ganita 3, 41–57 (1952).

Solutions of the problem of estimating the mean and the variance of a normal population, given a sample from that population after one-sided truncation, were obtained by Pearson and Lee (1908) who used the method of moments and by Fisher (1931) who derived the maximum likelihood estimates and proved the equivalence of the two methods. Both published tables of auxiliary functions needed for practical use of their methods. In the present paper the tables of Pearson-Lee and those of Fisher are re-computed for the argument proceeding by smaller intervals and over a wider range. In addition, the author proves inequalities on various functions related with the normal probability distribution, stating that at the time of writing the paper he was unaware of earlier proofs of some of these inequalities.
Z. W. Birnbaum (Seattle, Wash.).

Pillai, K. C. S. Some notes on ordered samples from a normal population. Sankhyā 11, 23–28 (1951).

Let $x_1, x_2, \cdots, x_n$ be an ordered sample of n from a normal population with zero mean and unit variance. Define $M_i=\frac{1}{2}(x_{n-i+1}+x_i)$, $W_i=\frac{1}{2}(x_{n-i+1}-x_i)$, the ith midrange and ith semi-midrange respectively. The author derives the joint distribution of (M_i, W_i) and the distributions of M_i and W_i as infinite series. The distribution of $T=M_1/W_1$, the distribution of the median for even sample size, and the distribution of the quotient of two independent sample ranges, $F'=W_1/W_1'$, are derived, all expressed as infinite series.

In Table I the author lists to 2 or 3 decimals the two-sided 5% and 1% values for T for $n=3(1)10$. Previously Walsh [Ann. Math. Statistics 20, 257–267 (1949); these Rev. 11, 191] had calculated approximate values of $\frac{1}{2}T$ for the 1%, 2%, 5%, and 10% levels for $n=3(1)10$. In Table III the author lists to 3 or 4 significant figures the upper 5% and 1% values of F' for all n_1, $n_2 \leqq 8$. Link [ibid. 21, 112–116 (1950); these Rev. 11, 446] expressed the distribution of F' by means of an integral and gave the lower $\frac{1}{2}$%, 1%, $2\frac{1}{2}$%, and 5% points usually to 2 significant figures for all n_1, $n_2 \leqq 10$, results readily transformed to upper percentage points. A comparison of these results indicates that Link's values for the 1% and 5% points are correct to ± 1 in the last figure. In Pillai's table of F' for $n_1=3$, $n_2=2$ replace 10.08 at the 5% level by 19.08, and apparently for $n_1=8$, $n_2=2$ at the 1% level, replace 169.0 by 159.0. For the 1% level differences indicate some error for either $n_1=6$, $n_2=2$ or $n_1=7$, $n_2=2$. The author concludes with a short table of the power of F' as compared with the usual F test for selected values of σ_1/σ_2 and (n_1, n_2). *L. A. Aroian.*

Walker, A. M. Note on sequential sampling formulae for a binomial population. J. Roy. Statist. Soc. Ser. B. 12, 301–307 (1950).

Consider the sequential sampling plan for testing the mean of a binomial distribution which adds 1 to an initial score h_2 for each non-defective item sampled and subtracts b for each defective item until the total score is either h_1+h_2 or 0. The author derives formulae for the operating characteristic and the first two moments of the average sample number of this plan in the case when b is a rational number, thus extending the work of Burman [Suppl. J. Roy. Statist. Soc. 8, 98–103 (1946); these Rev. 8, 395] who (among others; see the cited review) investigated the case when b is integral. *G. E. Noether* (Boston, Mass.).

Kazami, Akiko. Asymptotic properties of the estimates of an unknown parameter in stationary Markoff process. Ann. Inst. Statist. Math., Tokyo 4, 1–6 (1952).

Under regularity conditions analogous to those of Cramér [Mathematical methods of statistics, Princeton, 1946; these Rev. 8, 39] and using his idea of proof, the author shows that the maximum likelihood estimator is consistent and asymptotically efficient in the wide sense for the case of a stationary Markoff process. (Just as Cramér did, the author proves that a root of the likelihood equation has these properties. The possibility that the likelihood equation may have more than one root is not discussed.) If the process is Gaussian, the estimators obtained are asymptotically normally distributed. *J. Wolfowitz* (Ithaca, N. Y.).

Paulson, Edward. An optimum solution to the k-sample slippage problem for the normal distribution. Ann. Math. Statistics 23, 610–616 (1952).

Let O_i: $(x_{i1}, x_{i2}, \cdots, x_{in})$ $(i=1, 2, \cdots, k)$ be k random mutually independent samples, each drawn from the normal population $N(m_i, \sigma^2)$, with a common, but unknown variance σ^2. Let D_0 denote the decision that the k means are all

equal, and let D_j $(j=1, 2, \cdots, k)$ denote the decision that D_0 is incorrect and $m_j=\max(m_1, m_2, \cdots, m_k)$. The category Π_i is said to have slipped by an amount Δ $(\Delta>0)$ if $m_1=m_2=\cdots=m_{i-1}=m_{i+1}=\cdots=m_k$ and $m_i=m_1+\Delta$. This paper considers the problem how to find a statistical procedure for selecting one of the decisions $(D_0, D_1, \cdots, D_k)$ which will maximize the probability of making the correct decision when some category has slipped to the right, subject to certain restrictions, and the optimum solution is shown to be the procedure: (i) if $n(\bar{x}_M-\bar{x})/(\sum_{i=1}^k\sum_{\alpha=1}^n(x_{i\alpha}-\bar{x})^2)^{1/2}>\lambda_\alpha$, select D_M; (ii) otherwise select D_M, where $\bar{x}$ denotes the total sample mean, $\bar{x}_M$ the maximum of k sample means, and λ_α is a constant whose precise value is determined by the requirement that when all means are equal, D_0 should be selected with probability $1-\alpha$, and approximate approach is indicated to evaluate λ_α. *T. Kitagawa.*

Matusita, Kameo, and Akaike, Hirotugu. Note on the decision problem. Ann. Inst. Statist. Math., Tokyo 4, 11–14 (1952).

Let $\Omega=\{p_\nu\}$ be a set of density functions, and let p_0 be a density function not in Ω. Define

$$\rho(p_0, p_\nu)=\int[p_0(x)p_\nu(x)]^{1/2}dx.$$

Suppose $\sup_\nu \rho(p_0, p_\nu)=\rho<1$, $(n-1)\rho^k<\epsilon$, and there exist n functions in Ω, say $p_1, \cdots, p_n$, such that

$$\left\{x\left|\prod_{j=1}^k p_0(x_j)>\sup_{i=1,\cdots,n}\prod_{j=1}^k p_i(x_j)\right.\right\}=\left\{x\left|\prod_{j=1}^k p_0(x_j)>\sup_\nu\prod_{j=1}^k p_\nu(x_j)\right.\right\}=E_0,$$

say. Then $\int_{E_0}\prod_{j=1}^k p_0(x_j)dx_1\cdots dx_k>1-\epsilon$, and, for any ν, $\int_{E_0}\prod_{j=1}^k p_\nu(x_j)dx_1\cdots dx_k<\epsilon$. Several applications to testing hypotheses are given. *J. Wolfowitz* (Ithaca, N. Y.).

Sundström, Mauritz. Some statistical problems in the theory of servomechanisms. Ark. Mat. 2, 139–246 (1952).

The author deals with a great variety of statistical problems all of which occur in the theory of servomechanisms. Among these are the following: the determination of the argument of the transfer function $Y(j\omega)$ in an interval $(0, \omega_m)$ when $|Y(j\omega)|$ is given in this interval; the calculation of the inverse Laplace transform, assuming that the transform is known only in some interval $(0, \omega_m)$; the effect on the output of omitting input frequencies in a linear system; an investigation of the probability distributions of the input and output where the input is not necessarily stationary and where the mechanism is not necessarily linear; noise in linear systems; random errors in autocorrelation functions and spectral densities calculated from empirical data; and the influence of noise on the quantity of information. Although the paper contains an abundance of ideas, it must be said that in many instances the author has been content to justify his approximations merely by means of idealized examples. On the other hand, the problems considered are extremely complex and the approach developed by the author should be very useful to the workers in this field. *R. S. Phillips* (Los Angeles, Calif.).

Cox, D. R. Some recent work on systematic experimental designs. J. Roy. Statist. Soc. Ser. B. 14, 211–219 (1952).

Mathematical Biology

Tricomi, Francesco G. Distribuzione statistica dei batteri "duri a morire." Univ. e Politechnico Torino. Rend. Sem. Mat. 11, 21–34 (1952).

It is known that the application of antibiotics may result in the development of resistant strains of bacteria. This phenomenon could be explained by either one of the following hypotheses: (1) bacteria which were only slightly exposed to the antibiotic develop an immunity against it which is transmitted to their descendants; (2) spontaneous mutations make individual bacteria and their descendants resistant so that only the descendants of the bacteria which underwent mutation survive. An experiment was proposed by Delbruck which leads to the rejection of (1). The author constructs a probabilistic model for this experiment and develops it to the point where it becomes applicable to experimental data. *E. Lukacs* (Washington, D. C.).

Trucco, Ernesto. The smallest value of the axon density for which 'ignition' can occur in a random net. Bull. Math. Biophys. 14, 365–374 (1952).

It has been shown by Rapoport [same Bull. 14, 35–44 (1952); these Rev. 13, 763] that a net whose axon density is sufficiently high can be set into a state of continuous activity by an instantaneous stimulus of sufficient strength. The required density $A(h)$ is a function of the threshold h, assumed the same for all neurons. This paper describes the general properties of $A(h)$. *A. S. Householder.*

Rapoport, Anatol. Response time and threshold of a random net. Bull. Math. Biophys. 14, 351–363 (1952).

In a random net (of neurons) there is a constant probability that any given neuron will synapse with any other. Assuming that a particular response will occur when a given fraction of the neurons of the net are firing, the author studies the length of time required for the response following application of the stimulus (intensity-time relations) for the case of an instantaneous stimulus, and for that of a stimulus held constant. These relations resemble those deduced by Rashevsky [Mathematical biophysics, rev. ed., Univ. of Chicago Press, 1948] from the two-factor theory.

A. S. Householder (Oak Ridge, Tenn.).

Goodman, Leo A. On optimal arrangements in some social learning situations. Bull. Math. Biophys. 14, 307–312 (1952).

The author considers paratroopers learning to jump, in a situation where they are supposed to jump in some specified sequence, but some may refuse when this time comes. Let $s_{jk}(p)$ represent the probability that an individual will jump after he has seen j jump and k refuse, where $s_{00}(p)=p$. Let $n_{jk}=s_{jk}/s_{j-1,k}$ and $q_{jk}=f_{jk}/f_{j,k+1}$ where $f_{jk}=1-s_{jk}$. The main theorem is that if $n_{jk}(p)$ and $q_{jk}(p)$ are nonincreasing functions of p, then the sequence in which the best man jumps first, the second best second, etc., is optimal.

A. S. Householder (Oak Ridge, Tenn.).

Rapoport, Anatol, and Rebhun, Lionel I. On the mathematical theory of rumor spread. Bull. Math. Biophys. 14, 375–383 (1952).

Let $x_0=x(0)$ represent the fraction of a population from which a rumor starts, $x(n)$ the fraction who hear it not more than nth hand from the fraction x_0, then approximately in a large population $x(n+1)=1-(1-x_0)\exp[-ax(n)]$, where

s is constant. If $G(t)$ is the probability that an individual who hears the rumor will repeat it in less than t units of time, and $f(t)$ is the fraction having heard it by time t, then $f(t)=x_0+\sum_1^\infty[x(n)-x(n-1)]G_n(t)$, where G_n is the convolution of order n. The special case $G=1-\exp(-kt)$ is examined in detail. *A. S. Householder.*

TOPOLOGY

***Johansson, Ingebrigt. Present-day topology.** Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 55–60. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr.

Errera, Alfred. Une vue d'ensemble sur le problème des quatre couleurs. Univ. e Politecnico Torino. Rend. Sem. Mat. 11, 5–19 (1952).

Expository paper.

Infantozzi, Carlos A. Extensions of the theorems of Chittenden, Fréchet and Appert and of covering properties in (V) spaces. Bol. Fac. Ingen. Montevideo 4 (año 15), 409–419 (1951). (Spanish)

The paper discusses the covering axiom $B(\alpha,\beta)$: "every covering of the space by less than β open sets has a subcovering of less than α members," and its limiting variants.
R. Arens (Los Angeles, Calif.).

Morita, Kiiti. On bicompactifications of semibicompact spaces. Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. 4, 222–229 (1952).

A Hausdorff space R is called semibicompact [Zippin, Amer. J. Math. 57, 327–341 (1935)] if for $p\in R$ and a neighborhood U of p there is an open set V such that $p\in V\subset U$ and the boundary $B(V)$ of V is bicompact. The bicompactification of such a space was treated by Zippin and by Freudenthal [Ann. of Math. (2) 43, 261–279 (1942); these Rev. 3, 315] in case R has a countable basis. Those results are here relieved of the assumption of separability. The principal result is that if R is semibicompact, then R is completely regular and there is an essentially unique bicompact space S such that: (a) R is a dense subspace of S; (b) for $p\in S$ and a neighborhood U of p there is a set V, open in S, such that $p\in V\subset U$ and $B(V)\subset R$; (c) if S' is a bicompact space with properties (a), (b), then there is a continuous mapping of S' onto S leaving R invariant. The proof rests on the observation that if M denotes a finite covering of R by open sets with bicompact boundaries then the class $[M_\alpha]$ of all such coverings is a completely regular T-uniformity agreeing with the topology of R [Morita, Proc. Japan Acad. 27, 65–72, 130–137, 166–171 (1951); these Rev. 14, 68]. *L. W. Cohen* (Princeton, N. J.).

Morita, Kiiti. On the simple extension of a space with respect to a uniformity. IV. Proc. Japan Acad. 27, 632–636 (1951).

It is shown that: If $[\mathfrak{U}_\alpha|\alpha\in\Omega]$ is a regular uniformity on a space R agreeing with the topology, then the simple extension R^* of R [cf. Morita, same Proc. 27, 65–72, 130–137, 166–171 (1951); these Rev. 14, 68] with respect to $[\mathfrak{U}_\alpha]$ is characterized as a space S such that: (1) R is a subspace of S; (2) $[S-\overline{R-G}|G$ open in $R]$ is a basis for open sets in S; (3) each point of $S-R$ is closed; (4) $[S-\overline{R-U}\,|\,U\in\mathfrak{U}_\alpha]=\mathfrak{B}_\alpha$ is an open covering of S; (5) $[S(x,\mathfrak{B}_\alpha)|\alpha\in\Omega]$ is a neighborhood basis at $x\in S-R$. ($\bar{M}$ indicates closure in S.) If R is a regular T-space, is a dense subspace of a T-space S and each $x\in S-R$ is closed, then there is a regular T-uniformity $[\mathfrak{U}_\alpha]$ agreeing with the topology of R such that the corresponding simple extension R^* is homeomorphic with S by a mapping leaving the points of R fixed. If R is completely regular and S, T are simple extensions of R with respect to completely regular T-uniformities $[\mathfrak{U}_\alpha]$, $[\mathfrak{B}_\alpha]$, respectively, which consist of finite open coverings agreeing with the topology of R, then there is a continuous map f of S onto T such that $f(S-R)=T-R$ and $f(x)=x$ for $x\in R$ if and only if $[\mathfrak{U}_\alpha]$ contains a refinement of every covering in $[\mathfrak{B}_\alpha]$. A uniformity is said to agree strongly with the topology of R if $[S(S(x,\mathfrak{U}_\alpha),\mathfrak{U}_\beta)|\alpha,\beta\in\Omega]$ is a neighborhood basis for $x\in R$. In case $[\mathfrak{U}_\alpha]$ is a completely regular T-uniformity agreeing strongly with the topology of R a bicompactification of R is obtained of which R^* is a subspace and which is asserted to coincide with that of P. Samuel [Trans. Amer. Math. Soc. 64, 100–132 (1948); these Rev. 10, 54]. *L. W. Cohen* (Princeton, N. J.).

Padmavally, K. A characterization of minimally bicompact spaces. J. Indian Math. Soc. (N.S.) 16, 63–68 (1952).

A space is minimally bicompact if it is not possible to introduce additional sets as open without losing bicompactness. The property of such spaces here established as characteristic is this: for every limit point p of a closed set A there is a subset Z of A not containing p of which p is the sole complete limit point. It is also proved that the set P_t of points at which a bicompact topology t is minimally bicompact can be augmented by any scattered set A by introducing (possibly) more open sets affecting the topology only at A (retaining bicompactness). *R. Arens.*

Sugawara, Masahiro. On the metrizable condition. Proc. Japan Acad. 27, 625–626 (1951).

Let S be a neighborhood space topologized by a neighborhood basis $[U_\alpha(p)]$, $p\in S$, $\alpha\in A$ such that:

(1) $\bigcap_{\alpha\in A}U_\alpha(p)=[p]$;

(2) for $\alpha,\beta\in A$, $p\in S$, there is $\gamma=\gamma(\alpha,\beta;p)\in A$ such that $U_\gamma(p)\subset U_\alpha(p)\cap U_\beta(p)$; (3) for $\alpha\in A$, $p\in S$, there are $\lambda(p,\alpha)$, $\delta(p,\alpha)\in A$ such that if $U_{\delta(p,\alpha)}(q)\cap U_{\lambda(p,\alpha)}(p)\neq 0$, then $U_{\delta(p,\alpha)}(q)\subset U_\alpha(p)$. It is shown that if A is the set of natural numbers, then S is metrizable. It follows that a Hausdorff space S satisfying the first denumerability axiom and the condition (3′) obtained from (3) by putting $\lambda(p,\alpha)=\lambda(\alpha)$ for $p\in S$ is metrizable, thus answering affirmatively a question put by the reviewer [Duke Math. J. 5, 174–183 (1939); cf. Suzuki, Proc. Japan Acad. 27, 219–223 (1951); these Rev. 14, 69]. It is also shown that the first denumerability axiom, (1) and (3) are not sufficient for the metrization of S. *L. W. Cohen* (Princeton, N. J.).

Iwamura, Tsurane. Remarks on closed mapping and compactness. Nat. Sci. Rep. Ochanomizu Univ. 1, 6–8 (1951).

A set S with a closure operator $\bar{X}$ is called a space if: $\bar{O}=O$, $X\subset\bar{X}\subset\bar{\bar{X}}$, $\bar{X}\subset\bar{Y}$ if $X\subset Y$. Let f be a mapping of a space S into a space E and denote by $E^*(f)$ the class of sets $f^{-1}(p)\subset S$ for $p\in f(S)$. If f is a closed mapping of a space S into a space E such that the sets in $E^*(f)$ are compact, then S is compact. Suppose that S is a completely regular space

and that $\mathfrak{B}=[V_\alpha]$ is a uniform structure for S in the sense of A. Weil. A mapping f of such an S into a space E is called V-continuous if for $F_0 \in E^*(f)$ and $V_\alpha \in \mathfrak{B}$ there is some $V_\beta \in \mathfrak{B}$ such that if $F \in E^*(f)$ and $F \cap V_\beta(F_0) \neq 0$, then $F \subset V_\alpha(F_0)$. If an open mapping f of a completely regular space S onto a space E is V-continuous, then f is a closed mapping and consequently S is compact. *L. W. Cohen.*

Ball, B. J. Continuous and equicontinuous collections of arcs. Duke Math. J. 19, 423–433 (1952).

Let G be a collection of disjoint arcs in a metric space, and let G^* be the union of the elements of G. Suppose that G is a continuous collection, in the sense that the natural map of G^* onto G is both continuous and interior. Suppose further that G is equicontinuous, in the sense that for every positive number ϵ there is a $\delta>0$ such that if p, $q \in g \in G$, and $d(p, q)<\delta$, then the subarc pq of g is of diameter $<\epsilon$. Suppose finally that G^* is compact. The principal result is that (I) if G^* is a connected set in the plane, then there is a homeomorphism of the plane onto itself, throwing each element of G onto a linear interval. This requires 11 pages of argument, with a number of preliminary results, of which the following are typical: (II) If G^* is connected, and lies in the plane, then the union K of the end-points of the elements of G is the union either of two disjoint arcs or of two disjoint 1-spheres; and G^* is homeomorphic either to a 2-cell or to a closed plane annulus bounded by two concentric circles. (III) If G^* is a connected set, not necessarily in the plane, and K is as in (II), then K is closed, and has at most two components. *E. E. Moise.*

Block, H. D., and Cargal, Buchanan. Arbitrary mappings. Proc. Amer. Math. Soc. 3, 937–941 (1952).

Let X be a set of elements and $\mathfrak{N}=\{N\}$ a collection of nonempty subsets of X having a subcollection $N^1, N^2, \cdots$ such that if $x \in N \in \mathfrak{N}$, then $x \in N^k \subseteq N$ for some k. $N(x)$ denotes an N which contains x.

Repeat the above, replacing $X, \mathfrak{N}, N, x$ by $Y, \mathfrak{M}, M, y$, respectively. Let f be a correspondence which assigns to each x a nonempty subset of Y, denoted by $f(x)$. If $V\subseteq Y$, then $f^{-1}(V)$ is the set of all x such that $f(x)\cdot V\neq 0$. If $S\subseteq X$, S is nowhere dense if for each N there is an $N_1\subseteq N$ with $N_1\cdot S=0$. A set is exhaustible if it is the union of a countable collection of nowhere dense sets. The complement of an exhaustible set is a residual set. A point x is a point of concentrated inexhaustible f-approach if for each y in $f(x)$ and each $M(y)$ there is an $N(x)$ such that for every $N_\alpha \subseteq N(x)$, the set $[f^{-1}(M(y))]\cdot N_\alpha$ is inexhaustible (not exhaustible). Theorem: The points of concentrated inexhaustible f-approach form a residual set. Other results are given, but their statements require additional definitions. The authors state that all the results are applicable to an arbitrary operator on a separable Banach space, and that related results are true for linear operators on an arbitrary Banach space. The paper generalizes some results of H. Blumberg [Trans. Amer. Math. Soc. 24, 113–128 (1923); Duke Math. J. 11, 671–685 (1944); these Rev. 6, 205].
A. B. Brown (Flushing, N. Y.).

Koseki, Ken'iti. Über die Begrenzung eines besonderen Gebietes. III. Jap. J. Math. 21 (1951), 131–144 (1952).

[For parts I and II see these Rev. 7, 335; 10, 389; 11, 382.] Let r be a bounded set consisting of the common boundary of two plane domains G_1, G_2. The author proves that the points of r situated on a same prime end of G_1 constitute either a continuum of condensation in r, or else the sum of at most two indecomposable continua, or finally the sum of an indecomposable continuum and a continuum of condensation of r. This statement is somewhat complicated and the reviewer suggests that it may perhaps be replaced by a simpler one by making use of the notion of wing of a prime end [Ursell and Young, Mem. Amer. Math. Soc., no. 3 (1951); these Rev. 13, 55]. The author goes on to establish by means of prime ends a second theorem proved differently in the work of Kuratowski [Fund. Math. 12, 20–42 (1928)]. *L. C. Young* (Madison, Wis.).

Bundgaard, Svend. On a kind of homotopy in regular numbered complexes. Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 35–46 (1952).

The reviewer [J. London Math. Soc. 10, 21–25 (1935)] and F. Lannér [same Comm. 11 (1950); these Rev. 13, 58] enumerated the groups generated by reflections in the bounding hyperplanes of a finite simplex in n-dimensional spherical, Euclidean or hyperbolic space, showing that every such group has an abstract definition of the form

$$(r_\lambda r_\mu)^{n_{\lambda\mu}}=1 \qquad (\lambda, \mu=0, 1, 2, \cdots, n)$$

where $n_{\lambda\mu}$ is finite for each choice of λ, μ and equal to 1 when $\lambda=\mu$. The present author extends this idea by considering analogous simplicial subdivisions of further pseudomanifolds. He declares that the group has the property (M) if every relation between the generators is a consequence of the above system of relations, and every relation between some of the generators is a consequence of the subsystem in which only these generators occur. He proves that the group has the property (M) if and only if the pseudomanifold itself is simply connected while also the stars of simplexes surrounding each vertex, edge, $\cdots$, and $(n-3)$-cell are simply connected. *H. S. M. Coxeter* (Toronto, Ont.).

Toda, Hirosi. On the homotopy groups of spheres. Kôdai Math. Sem. Rep. 1952, 93–94 (1952).

The main result of this note is the determination of the n-dimensional homotopy group of the r-dimensional sphere $\pi_n(S^r)$ for the following pairs (n, r): $(r+3, r)$, $(r+4, r)$, $(s+5, s)$, where $r\geqq 1$ and $s\geqq 5$. In addition it is asserted that $\pi_{n+6}(S^n)$ is a group of order 2 or 4 for $n\geqq 8$, and $\pi_{n+7}(S^n)$ is the direct sum of a cyclic group of order 15 and a 2-primary component. No proofs are given in this brief note, although the statements of some preliminary lemmas are given. The author asserts that his results depend strongly on the homology theory of abelian groups of Eilenberg and MacLane [Proc. Nat. Acad. Sci. U. S. A. 36, 443–447, 657–663 (1950); 37, 307–310 (1951); these Rev. 12, 350, 520; 13, 151]. Use is also made of recent results of J. P. Serre [Ann. of Math. (2) 54, 425–505 (1951); these Rev. 13, 574]. However, the author is apparently unaware of two more recent notes of Serre [C. R. Acad. Sci. Paris 234, 1243–1245, 1340–1342 (1952); these Rev. 13, 675] in which all of the above listed results on homotopy groups of spheres, with the exception of those on $\pi_{n+6}(S^n)$ and $\pi_{n+7}(S^n)$, are announced without proof. Moreover, Serre determines the structure of $\pi_8(S^3)$ and $\pi_9(S^3)$. *W. S. Massey.*

Toda, Hirosi. Some relations in homotopy groups of spheres. J. Inst. Polytech. Osaka City Univ. Ser. A. Math. 2, 71–80 (1952).

It was proved by H. Freudenthal [Compositio Math. 5, 299–314 (1937)] that the suspension homomorphism E: $\pi_n(S^r)\to\pi_{n+1}(S^{r+1})$ is an isomorphism onto if $n<2r-1$.

More recently, G. W. Whitehead showed that the kernel of the suspension homomorphism E is the cyclic subgroup generated by Whitehead products if $n=2r-1$ [cf. Ann. of Math. (2) **51**, 192–237 (1950); these Rev. **12**, 847]. In this paper, the author calculates certain special Whitehead products in homotopy groups of spheres, and proves that in certain cases the suspension homomorphism E is not an isomorphism. Among these may be mentioned the cases $n=r+4$ $(r=2, 4, 5)$ and $n=r+5$ $(r=2, 4, 5, 6)$.

W. S. Massey (Providence, R. I.).

Frenkel, Jean. Sur une classe d'espaces fibrés analytiques. C. R. Acad. Sci. Paris **236**, 40–41 (1953).

This concerns the analytic homotopy in the theory of complex analytic fibre spaces. Let M, G be two complex Lie groups. Suppose that G acts on M as a group of automorphisms such that the mapping: $G\times M\to M$, defined by $(y, x)\to y(x)$, $y\in G$, $x\in M$, is analytic. Each principal fibre space g with base X and fibre G gives then a fibre space M^g over X with fibre M and structure group G. Assuming X to be a Stein manifold, the author proves the following. (i) If M, G are connected and M solvable, then there is a 1-1 correspondence between the ordinary homotopy classes of sections of M^g and the analytic homotopy classes of analytic sections. (ii) If M is solvable, then the ordinary equivalence classes of principal fibre spaces over X with fibre M are in 1-1 correspondence with the analytic equivalence classes of analytic principal fibre spaces over X with fibre M.

H. C. Wang (Auburn, Ala.).

Rohlin, V. A. New results in the theory of four-dimensional manifolds. Doklady Akad. Nauk SSSR (N.S.) **84**, 221–224 (1952). (Russian)

Two smooth, compact, oriented, k-dimensional manifolds M, N are called homologous, $M\sim N$, if there is a smooth, compact, oriented, $(k+1)$-dimensional manifold whose boundary is $M-N$ ($M-N$ denotes the conjunction of M and $-N$, $-N$ denotes N with orientation reversed). The resulting homology classes are "added" by conjoining representative manifolds; the additive group $\mathfrak{O}^k$ thus obtained is called the k-dimensional homology group. That $\mathfrak{O}^1=0$ and $\mathfrak{O}^2=0$ express familiar facts. The author has previously shown [same Doklady (N.S.) **81**, 355–357 (1951; these Rev. **14**, 72] that $\mathfrak{O}^3=0$. In the present paper it is shown that $\mathfrak{O}^4$ is an infinite cyclic group and that a generator is represented by the oriented complex projective plane P_4. The Pontrjagin characteristic number X_{22} [Mat. Sbornik N.S. **21**(63), 233–284 (1947); these Rev. **9**, 243] defines a homomorphism of $\mathfrak{O}^4$ onto the integral multiples of 3. Furthermore, $X_{22}(M^4)=3\sigma(M^4)$, where $\sigma(M^4)$ is the signature of the integral quadric form (x, x), and (x, y) denotes the intersection number of elements x, y of the 2-dimensional Betti group of M^4.

For non-orientable k-dimensional manifolds an analogous group $\mathfrak{N}^k$ (whose elements are of order 2) is defined. $\mathfrak{N}^1=\mathfrak{N}^3=0$, $\mathfrak{N}^2$ and $\mathfrak{N}^4$ are cyclic of order 2; $\mathfrak{N}^2$ is generated by the projective plane P^2 and $\mathfrak{N}^4$ by the complex projective plane P^4.

In previous papers [Doklady Akad. Nauk SSSR (N.S.) **80**, 541–544; **81**, 19–22 (1951); these Rev. **13**, 674] the author had incorrectly calculated $\pi_6(S^3)$; this mistake he now rectifies, and calculates $\pi_{n+3}(S^n)$ for all $n\geq 3$. The results agree with, and were anticipated by, results of Massey, G. W. Whitehead, Barratt, Paechter and Serre [these Rev. **13**, 674, 675].

Next the author states that $\sigma(M^4)\equiv 0 \pmod{16}$ whenever $x^2(M^4)=0$, where x^2 is the 2-dimensional characteristic cohomology class of M^4. From this and results announced by Wu [C. R. Acad. Sci. Paris **230**, 508–511 (1950); these Rev. **12**, 42] he deduces that the quadratic form (x, x) of a simply connected closed 4-dimensional manifold M^4 cannot be an even-valued positive definite form of rank 8 with discriminant 1. Since such forms were constructed by Korkin and Zolotarev [Math. Ann. **6**, 366–389 (1873)] it follows that an M^4 is not associated with every arithmetical type of quadratic form.

The author's result about $\mathfrak{O}^4$ was anticipated, in part, by R. Thom [Colloque de Topologie, Strasbourg, 1951, no. V; these Rev. **14**, 492].

R. H. Fox (Princeton, N. J.).

Čogošvili, G. S. On the fundamental homomorphisms of duality. Akad. Nauk Gruzin. SSR. Trudy Mat. Inst. Razmadze **18**, 1–52 (1951). (Russian. Georgian summary)

The author develops another extension of the Alexander-Kolmogorov duality theory [see these Rev. **12**, 846 for an earlier generalization as well as references to previous work by the author, by Alexandrov, and by Kaplan]. Here, certain natural homomorphisms of homology groups of an "admissable subset" A of a space R into the groups of the complementary subset $B=R-A$ are no longer necessarily isomorphisms. However, the homomorphisms are "good enough so that some properties of B can be deduced from the structure of A." Furthermore, when R has the appropriate properties and A is a closed subset of R, the homomorphisms become the isomorphisms of the Alexander-Kolmogorov theory.

Two general situations are encompassed. Either the space R is normal and A is closed or R is completely normal and conditions on A are "relaxed" a little. Thus, A is to be "bicompactly immersed" in R and its frontier a (points of A not inner-points of A) is to be of the type of a difference of two closed sets. The second condition is equivalent to the existence of an open set O in R such that $a\subset O$ and a is closed in O. Then O is called a "neighborhood of closedness" for the set a. The first condition means that if U is any relatively open subset of A whose closure is bicompact, then there is an open V in R whose closure is bicompact and such that $U=V\cap A$.

The author uses many classes of "inner" and "outer" cycles and cocycles, defined by coverings of Alexandrov types for the sets A and B, as well as for associated sets a (frontier), $\bar{A}$ (the closure), and B_i the inner-points of B. The principal new results concern the existence and nature of homomorphic maps of homology groups based on these classes of cycles.

L. Zippin (Flushing, N. Y.).

Hu, Sze-Tsen. Cohomology relations in spaces with a topological transformation group. Nagoya Math. J. **5**, 113–125 (1953).

Let Q be a topological transformation group acting on the left on a space X, B the orbit space, and $p: X\to B$ the projection. For any coefficient group G, the author constructs a group $H_w^n(X, G)$ and embeds the homomorphism $p^+: H^n(B, G)\to H^n(X, G)$ (Alexander-Spanier cohomology) in an exact sequence

$$\cdots\to H^n(B)\to H^n(X)\to H_w^n(X)\to H^{n+1}(B)\to\cdots.$$

For each $x\in X$, a homomorphism $k(x): H_w^n(X, G)\to H^n(Q, G)$ is also constructed; if Q is compact and x, x' are contained in a connected compact set, it is shown that $k(x)=k(x')$.

[Reviewer's remark: The author does not note that his $H_w^n(X)$ is essentially the group $H^{n+1}(C, X)$, where C is the mapping cylinder of p. If Ξ is the orbit of x, $\pi: Q\to\Xi$ the map $q\to qx$, D the mapping cylinder of $p|\Xi$, and $j: (D, \Xi)\to(C, X)$ the injection, then $\delta: H^n(\Xi)\approx H^{n+1}(D, \Xi)$ and the author's $k(x)$ is then essentially the homomorphism $\pi^+\delta^{-1}j^+$.]

J. Dugundji (Princeton, N. J.).

Torres, Guillermo. On the Alexander polynomial. Ann. of Math. (2) 57, 57–89 (1953).

Using the free calculus, one may associate to any finitely generated group G a unique principal ideal in the integral group ring JH of the Betti group H of G; the reviewer has called a generator $\Delta(t_1, \cdots, t_n)$ of this ideal the Alexander polynomial of G. By the Alexander polynomial of a knot $X=X_1\cup\cdots\cup X_\mu$ of multiplicity μ is understood the Alexander polynomial of the fundamental group of the complement of X; for $\mu=1$ this is identical with the polynomial defined by Alexander [Trans. Amer. Math. Soc. 30, 275–306 (1928)].

It was shown by Seifert [Math. Ann. 110, 571–592 (1934)] that the Alexander polynomials $\Delta(t)$ of knots of multiplicity $\mu=1$ are characterized by the following two properties: (1_1) $\Delta(1/t)=t^{2h}\Delta(t)$, (2_1) $\Delta(1)=\pm1$. The principal object of the paper at hand is to find analogous properties for the polynomials of knots of multiplicity $\mu>1$. The properties found are the following:

$$(1_\mu)\quad \Delta(1/t_1, \cdots, 1/t_\mu)=(-1)^{\mu}t_1^{n_1}\cdots t_\mu^{n_\mu}\Delta(t_1, \cdots, t_\mu);$$

$$(2_\mu)\quad \Delta(t_1, \cdots, t_{\mu-1}, 1)=\frac{t_1^{l_{1,\mu}}\cdots t_{\mu-1}^{l_{\mu-1,\mu}}-1}{(t_1\cdots t_{\mu-1}-1)^{\theta_\mu}}\Delta(t_1, \cdots, t_{\mu-1}),$$

where $\Delta(t_1, \cdots, t_{\mu-1})$ is the polynomial of the knot $X_1\cup\cdots\cup X_{\mu-1}$, l_{ij} denotes the linking number of the components X_i and X_j, and $\theta_\mu=0$ or 1 according as $\mu>2$ or $\mu=2$. (An interesting consequence of (2_2) is that $\Delta(1, 1)=\pm l_{12}$.)

The properties (1_μ) are geometrical, i.e., they are not true for the polynomial of an arbitrary group, and they are also quite hard to prove. (The author has, however, subsequently found simpler proofs, publication of which will ensue.) The properties (2_μ) are not so hard to prove; in fact (1_1), first proved by Alexander [loc. cit.], is not even a geometrical property, as it depends only on the fact that H is of rank 1.

Noteworthy features of the development of this paper: the generalization of the Seifert diagram to the case $\mu>1$; the generalization of the concept of genus of a knot to the case $\mu>1$; a lower bound for the genus in terms of μ and the degree of the polynomial $\Delta(t, \cdots, t)$; characterization of the polynomials $\Delta(t, \cdots, t)$. The polynomials $\Delta(t_1, \cdots, t_\mu)$, $\mu>1$, are, however, not themselves characterized; this remains an unsolved problem.

In the final chapter certain results of Seifert [Quart. J. Math., Oxford Ser. (2) 1, 23–32 (1950); these Rev. 11, 735] about polynomials of "knots within knots" are generalized from the case $\mu=1$ to the case $\mu>1$. This result, which is somewhat complicated to describe, has been subsequently generalized by the reviewer to a theorem about Alexander polynomials of 3-dimensional manifolds that contains in particular the reviewer's recently announced topological classification of the lens space.

R. H. Fox (Princeton, N. J.).

GEOMETRY

Thébault, Victor. On the skew quadrilateral. Amer. Math. Monthly 60, 102–105 (1953).

Thébault, Victor. Sur des plans associés à un tétraèdre. Ann. Soc. Sci. Bruxelles. Sér. I. 66, 111–118 (1952).

Petrov, G. On conditions for construction of a triangle. Časopis Pěst. Mat. 77, 77–92 (1952). (Czech)

If a, b, c are the sides of a triangle, and u_a the bisector of the angle opposite the side a, we have the known relation

$$u_a^2(b+c)^2=bc[(b+c)^2-a^2].$$

The author denotes a, u_a, b, c by x, y, z, t, and considers (x, y, z, t) as the rectangular homogeneous coordinates of a point in an extended three-dimensional Euclidean space. A discussion of the surface

$$f(x, y, z, t)\equiv y^2(z+t)^2-zt[(z+t)^2-x^2]=0$$

together with the inequalities

$$x+z-t>0,\quad x-z+t>0,\quad -x+z+t>0$$

yields a necessary and sufficient condition for the existence of a triangle when a, b, u_a are given. This method of analysis is applied to a considerable number of similar problems involving medians, altitudes, the circumradius, the inradius, and combinations of these elements. *N. A. Court.*

Tallqvist, Hj. Ort konstanter Summe oder Differenz der Tangentenlängen zu zwei Kreisen. Soc. Sci. Fenn. Comment. Phys.-Math. 15, no. 2, 20 pp. (1951).

The author discusses in great detail the various special cases of the locus of a point moving so that the sum or difference of its tangential distances from two fixed circles is constant. *S. B. Jackson* (College Park, Md.).

Tallqvist, Hj. Einige geometrische Örter. Soc. Sci. Fenn. Comment. Phys.-Math. 15, no. 5, 28 pp. (1951).

The author considers in detail eight elementary locus problems, all of which lead to families of conics. No techniques are used beyond elementary analytic geometry.

S. B. Jackson (College Park, Md.).

Tallqvist, Hj. Über Örter gleicher Gesichtswinkel in bezug auf zwei Gegenstände. Soc. Sci. Fenn. Comment. Phys.-Math. 16, no. 7, 13 pp. (1952).

As indicated in the title, the paper considers by means of elementary analytic geometry a few special cases of loci arising by asking that the angles subtended by two given objects in the plane are equal. The cases considered include two ellipses and an ellipse and parabola in special positions and some of their degenerate cases. *S. B. Jackson.*

Tallqvist, Hj. Auf zwei Kreise sich beziehende Probleme. Soc. Sci. Fenn. Comment. Phys.-Math. 16, no. 8, 10 pp. (1952).

A series of exercises in elementary analytic geometry concerned with two non-concentric circles.

S. B. Jackson (College Park, Md.).

Tallqvist, Hj. Einige auf eine Gerade und einen Kreis sich beziehende Aufgaben. Soc. Sci. Fenn. Comment. Phys.-Math. 16, no. 9, 13 pp. (1952).

Elementary problems in analytic geometry on loci related to a circle and a line or a circle and a line segment.

S. B. Jackson (College Park, Md.).

Tallqvist, Hj. Geometrische Örter bei einem Kegelschnitt. Soc. Sci. Fenn. Comment. Phys.-Math. 16, no. 10, 11 pp. (1952).

For a given conic section the author considers the following loci: the locus of points having fixed distance from their polars, the locus of points at fixed tangential distance from the conic, and the locus of points at a fixed normal distance from the conic (i.e., the parallel curves to the conic). The methods are entirely elementary. *S. B. Jackson.*

Beskin, N. M. Analog of the theorem of Pohlke-Schwarz in central axonometry. Doklady Akad. Nauk SSSR (N.S.) 50, 41–44 (1945). (Russian)

A detailed version of this paper appeared in Mat. Sbornik N.S. 19(61), 57–72 (1946); these Rev. 8, 220.

Četveruhin, N. F. On a fundamental theorem of axonometry in central projection. Doklady Akad. Nauk SSSR (N.S.) 50, 75–76 (1945). (Russian)

Given a plane Desargues configuration

$$(F) = (Oxyz, ABC, PQR)$$

composed of two triangles ABC, PQR having O for center of perspectivity, E. Kruppa established a necessary and sufficient condition for (F) to be a central projection of a configuration in space $(F') = (O'x'y'z', A'B'C', P'Q'R')$ consisting of three mutually orthogonal axes $O'x'$, $O'y'$, $O'z'$, their unit points A', B', C' $(O'A' = O'B' = O'C')$, and their points at infinity P', Q', R' [S.-B. Akad. Wiss. Wien. Math.-Nat. Kl. 119, Abt. IIa, 487–506 (1910)]. The author offers an elementary proof of Kruppa's proposition.

N. A. Court (Norman, Okla.).

★**Lietzmann, W. Anschauliche Einführung in die mehrdimensionale Geometrie.** Verlag von R. Oldenbourg, München, 1952. 220 pp. (1 plate).

After a literary introduction, referring to Maeterlinck, Wells, Verne, Dunne, and others, the author shows how Hilbert's axioms can be extended to n dimensions. He then describes the general simplex, parallelotope, prism, pyramid, and cross polytope ("Oktaederpolytop"). He proves Euler's formula and its n-dimensional generalization. He gives an analytic treatment of orthogonality and parallelism, mensuration, congruence, and similarity. He proves that there cannot be more than six convex regular polytopes in four dimensions. As an excuse for not attempting to describe the 600-cell $\{3, 3, 5\}$ and the 120-cell $\{5, 3, 3\}$, he quotes a remark of Schoute to the effect that the numbers of vertices, edges, faces, and cells can be computed by elementary means only for the four simpler polytopes. This shows that the author never read D. M. Y. Sommerville's Geometry of n dimensions [Methuen, London, 1929], nor the reviewer's Regular polytopes [Methuen, London, 1948; these Rev. 10, 261]. More surprisingly, he is ignorant of the work of Schläfli [Gesammelte mathematische Abhandlungen, Birkhäuser, Basel, 1950; these Rev. 11, 611]. On the other hand, he refers to Schoute [Mehrdimensionale Geometrie, Göschen, Leipzig, 1902, 1905] and to Hilbert and Cohn-Vossen [Anschauliche Geometrie, Springer, Berlin, 1932; Geometry and the imagination, Chelsea, New York, 1952; these Rev. 13, 766].

The book is well written in an elementary fashion, beautifully printed, and illustrated with 157 diagrams and one plate of photographs of models. *H. S. M. Coxeter.*

★**Smogorževskiĭ, A. S. Geometričeskie postroeniya v ploskosti Lobačevskogo.** [Geometric constructions in the Lobačevskiĭ plane.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 191 pp. 5.20 rubles.

A very well-written book with many hitherto unpublished constructions. It is self-contained inasmuch as all the results of hyperbolic geometry needed in the book are derived in the first three chapters which deal with: I) Models of the hyperbolic plane (pseudo-sphere, Klein model, Poincaré model); II) Relation between the models, trigonometry, equidistant curves, limit circles; III) Radical axes of circle, inversion.

In chapter IV the simple operations are defined. The simplicity coefficient S of a construction is the number of simple operations required. The basic constructions are discussed, including new constructions for the distance belonging to a given angle as parallel angle ($S=30$), the triangle from its angles ($S=82$), and the regular 17-gon with $S=217$ instead of previously 280.

Chapter V deals with the question of constructability; for instance, a segment can in general not be trisected. It is shown that all constructions feasible with straight edge and compass can also be effected with the straight edge and a ruler drawing limit circles (properly defined), or the straight edge and one ruler whose edges are a straight line and a curve equidistant to it. Chapter VI deals with construction by means of a straight edge and one fixed figure. Three examples will suffice to indicate the trend: all constructions feasible with straight edge and compass can be accomplished if the given figure is 1) a circle with its center and a pair of parallel lines, 2) a line, a curve equidistant to it, and a pair of lines parallel to each other but not to the given line, 3) two limit circles tangent to each other. Chapter VII establishes the result corresponding to Mascheroni's in the euclidean case, namely, that all constructions feasible by straight edge and compass, can be performed by using instruments drawing circles, limit circles, and equidistant curves (but not straight lines), provided a line is considered known if two of its points are. Chapter VIII deals with the quadrature of the circle, i.e., the construction of an equiangular rhombus with the same arc as a given circle, and the converse problem. These problems can be solved for certain values of the radius or angle. The final chapter IX gives other solutions, mostly of historical importance, to certain of the previously treated problems, but treats also some new questions, for instance, Apollonius' problem.

H. Busemann (Los Angeles, Calif.).

Kagan, V. F. The development of interpretations of non-Euclidean geometry. Introductory considerations and the first development of interpretations of the geometry of Lobačevskiĭ. Moskov. Gos. Univ. Učenye Zapiski 148, Matematika 4, 3–8 (1951). (Russian)

Polozkov, D. P. A study of generalized interpretations of the plane geometry of Lobačevskiĭ and its geodesic lines. Moskov. Gos. Univ. Učenye Zapiski 148, Matematika 4, 9–14 (1951). (Russian)

If the interior I of the euclidean unit circle is metrized as the Klein model of hyperbolic geometry, then any topological mapping ϕ of I on itself yields another "model". In hyperbolic polar coordinates r, θ with the center of I as pole, special ϕ are given by $r' = \alpha r$, $\theta' = \theta$; for $\alpha = 2$ the Poincaré model is obtained. The first paper studies the models for integral α, the second for arbitrary real α. They are of little

interest for $\alpha \neq 1, 2$ since they lack both the projective character of Klein's model and the conformality of Poincaré's. Also, the reader derives the results almost as quickly for himself as he reads them. *H. Busemann.*

Yablonovskiĭ, A. V., and Yampol'skiĭ, V. G. Some special questions relating to interpretations of the plane geometry of Lobačevskiĭ. Moskov. Gos. Univ. Učenye Zapiski **148**, Matematika **4**, 15–20 (1951). (Russian)

This paper is a direct continuation of the paper reviewed above. A class of transformations of the absolute unit circle of type

$$R_\alpha(r)=\frac{(1+r)^{1/\alpha}-(1-r)^{1/\alpha}}{(1+r)^{1/\alpha}+(1-r)^{1/\alpha}}$$

and the corresponding geometrical interpretations are considered. Some results are not essentially different from B. F. Kagan's interpretation of Lobachevsky geometry. *H. A. Lauwerier* (Amsterdam).

Meĭler, N. A. The construction of a system of models of the elliptic plane in the Euclidean plane. Moskov. Gos. Univ. Učenye Zapiski **148**, Matematika **4**, 21–29 (1951). (Russian)

This paper extends the ideas of the preceding three papers to the elliptic plane by also introducing a parameter α into Klein's model for the elliptic plane. Again, for $\alpha=2$ Poincaré's model is obtained. The character of a "model" is largely lost inasmuch as the mapping corresponding to α ceases to be one-to-one, in fact is $\aleph_0$-to-$\aleph_0$ for irrational α, finite-to-finite for rational α, and 1-to-n for integral α. *H. Busemann* (Los Angeles, Calif.).

Smogorževskiĭ, A. S. On some geometric constructions in the hyperbolic and Euclidean planes. Doklady Akad. Nauk SSSR (N.S.) **50**, 61–63 (1945). (Russian)

A construction of a hyperbolic triangle from its angles, and a construction in the euclidean plane by use of the compass alone of x when a, b, c are given segments and $a:b=c:x$. *H. Busemann* (Los Angeles, Calif.).

Bompiani, Enrico. Sulle geometrie non-euclidee. Archimede **4**, 89–97, 143–147, 228–235 (1952); **5**, 9–16 (1953). Expository paper.

Fulton, C. M. A different approach to the non-Euclidean geometries. Amer. Math. Monthly **60**, 7–11 (1953).

It is shown that a 3-space in which the usual axioms of incidence, order, congruence, and continuity hold, together with the ordinary spherical trigonometry, must be hyperbolic, Euclidean, or a limited region of elliptic space. The functional equations used in the Gérard development of the plane non-Euclidean trigonometries [cf. J. L. Coolidge, The elements of non-Euclidean geometry, Oxford, 1909, p. 52] are derived from the present hypotheses, and the three geometries correspond respectively to the three solutions of the equations. *A. J. Hoffman* (Washington, D. C.).

Morgantini, Edmondo. Su una relazione di armonia fra i triangoli del piano proiettivo complesso. Ann. Triestini. Sez. 2. (4) **5(21)** (1951), 5–33 (1952).

In der Transformation

$$y_1:y_2:y_3=(x_1^2-lx_2x_3):(x_2^2-mx_3x_1):(x_3^2-nx_1x_2)$$

seien x_1, x_2, x_3 projektive Koordinaten der Ebene und A_1, A_2, A_3 die Ecken des Koordinaten-Dreiecks (A). Die Kegelschnitte $x_1^2-lx_2x_3=0$, $x_2^2-mx_3x_1=0$, $x_3^2-nx_1x_2=0$ schneiden sich genau dann in drei Punkten B_1, B_2, B_3, wenn $lmn=1$ ist. Sind $(x_1:x_2:x_3)$ die Koordinaten von B_1, so sind $(\epsilon x_1:\epsilon^2x_2:x_3)$ und $(\epsilon^2x_1:\epsilon x_2:x_3)$ mit $\epsilon^3=1$ die Koordinaten von B_2 und B_3. Die Zuordnung von (A) zu den ∞^2 Dreiecken (B) und deren Konfiguration wird vom Standpunkt der projektiven Abbildungen und der quadratischen Abbildungen der Ebene auf sich genauer untersucht. Die Dreiecke (B) heissen zu (A) harmonisch, denn in jedem (B) ist eine Seite harmonische Polare ihrer Gegenecke bezüglich (A). Es gibt genau zwei zueinander harmonische Dreiecke (C), (D), die zueinander harmonisch und zu jedem von zwei zueinander harmonischen Dreiecken (A), (B) harmonisch sind. *R. Moufang* (Frankfurt a.M.).

Laman, G. Distance geometry and Boolean algebras. Simon Stevin **29** (1951/52), 83–91 (1952).

As an introduction to the next paper by D. Ellis [see the paper reviewed below] the author gives a neat and lucid exposition of some of the basic notions and propositions used in abstract (algebraic) distance geometry: Groups and derived notions (groupoids, semigroups, . . .), Boolean algebras and their interpretation as Boolean rings, betweenness in general lattices, congruences and motions in autometrized Boolean algebras. [Remark by the reviewer: Theorems III–VII become evident if the "intrinsic" distance (discrepancy) $d(a, b)$ is written as $b-a=a-b$ in Stone's notation. The group of motions of the Boolean algebra simply becomes the group of translations of its Abelian additive group.] *Chr. Pauc* (Nantes).

Ellis, David. Notes on abstract distance geometry. III. On self-congruences of metroids. Simon Stevin **29** (1951/52), 92–95 (1952).

This note continues a series of publications [see these Rev. **13**, 270, 377, 970] devoted to abstract distance geometry. Distance-theoretic properties of a ground space are correlated with algebraic properties of its metroid. We reproduce here the Main Theorem: In a metroid M which is a semigroup with a unit e, the following propositions are pairwise equivalent: (1) There exists a motion sending e into any element of M. (2) The group of motions of M is simply transitive. (3) M has the property of free mobility. (4) All elements of M are nilpotent. (5) M is a subgroup of the additive group of a Boolean ring. The pattern for the proof is $(5)\rightarrow(4)\rightarrow(3)\rightarrow(2)\rightarrow(1)\rightarrow(5)$. The proof of the last implication rests on Moisil's theorem: An Abelian group of nilpotents is a subgroup of the additive group of a Boolean ring. The other proofs consist in straightforward manipulations. *Chr. Pauc* (Nantes).

Convex Domains, Extremal Problems, Integral Geometry

Gaddum, Jerry W. A theorem on convex cones with applications to linear inequalities. Proc. Amer. Math. Soc. **3**, 957–960 (1952).

The paper investigates the existence of solutions of linear homogeneous inequalities (1) $Ax\geq 0$ and (2) $Ax \geq 0$ (meaning $Ax\geq 0$ but not $Ax=0$), where A is a matrix, x a vector. The author first states a geometric theorem. If A is a convex cone in a complete inner product space and A^* is its polar, then $A\cap A^*=0$ if and only if A is a linear subspace. The proof given applies only under the additional hypothesis

that A be closed. Theorem 3.1: the system (2) has a solution if and only if $A\cap A^*\neq 0$ [here A represents both the matrix and the cone generated by its row vectors]. The proof, we are told, follows from Theorem 2.2 which, however, is not to be found in the paper. Theorem 2.1 was probably intended. Theorem 3.2: In order that (2) have a solution it is necessary and sufficient that $AA'y\geq 0$ have a non-negative solution. The final two theorems give conditions on the solvability of $AA'y\geq 0$. Theorem 3.3: $AA'y\geq 0$ has a solution if and only if $AA'y=0$ has no positive solution. Theorem 3.4: A necessary condition that $AA'y\geq 0$ have a solution is that $-AA'y=0$ have no positive solution. Since the solutions of $AA'y=0$ and $-AA'y=0$ are identical, 3.4 is a trivial restatement of part of 3.3 (unless this is another typographical error). *D. Gale* (Providence, R. I.).

Fiedler, Miroslav. Solution of a problem of Prof. E. Čech. Časopis Pěst. Mat. 77, 65–75 (1952). (Czech)

E. Cech proposed the problem of characterizing the sets in the (x, y)-plane which can be represented by an equation of the form

$$\sum_{r=1}^{n}|a_rx+b_ry+c_r|+ax+by+c=0.$$

The a_r, b_r, c_r, a, b, c are arbitrary real numbers. The author solves the problem by proving the following theorem: Call an intersection of a finite number of closed half-planes a K-set. Then the sets S which can be described by an equation of the above form are of one of the following two types: (a) S is a K-set; (b) S is the boundary of a K-set containing an interior point. The only sets to be excluded here are the boundaries of ordinary triangles and the boundaries of infinitely extended triangles with two parallel half-planes as part of the boundary. *C. Loewner* (Stanford, Calif.).

Van Heijenoort, John. On locally convex manifolds. Comm. Pure Appl. Math. 5, 223–242 (1952).

Let f be a mapping of a two-dimensional manifold M in E^3 with the following properties: 1) f is locally topological; 2) f is locally convex, i.e., every point $p\in M$ has a neighborhood N_p such that $f(N_p)$ lies on the boundary of a closed convex set K_p in E^3; 3) for at least one point $p\in M$ the body K_p has at $f(p)$ a supporting plane intersecting K_p only at $f(p)$. The metric in E^3 induces a length for curves in M, and any two points of M can be connected by curves of finite length. The last requirement is 4) M is complete if the distance pq on M is defined as the greatest lower bound of the lengths of all curves connecting p and q. It is proved that $f(M)$ is the (complete) boundary of a convex set in E^3 with interior points; hence M itself is homeomorphic to a sphere or to a plane. This theorem was proved previously by Hadamard for compact M and by Stoker for noncompact M under the assumption that M has positive Gauss curvature. Also, as the author observes, the theorem is contained in a result of A. D. Alexandrov, whose methods have deeper aims and are correspondingly more complicated. In addition, the present proof can be extended to $(n-1)$-dimensional M mapped in E^n, whereas Alexandrov's methods are typically two-dimensional. *H. Busemann* (Los Angeles, Calif.).

Pogorelov, A. V. On the unique determination of infinite convex surfaces. Učenye Zapiski Har'kov. Gos. Univ. 28, Zapiski Naučno-Issled. Inst. Mat. Meh. i Har'kov. Mat. Obšč. (4) 20, 53–60 (1950). (Russian)

Let F be a three times continuously differentiable complete open convex surface in E^3 with positive Gauss curvature. F can be represented in the form $z=f(x, y)>0$. Denote by $\psi(h)$ the integral curvature of the part $z\leq h$ of F and by $l(h)$ the length of the intersection of $z=h>0$ with F. If $[2\pi-\psi(h)]/l(h)\to 0$ for $h\to\infty$, then any three times differentiable surface intrinsically isometric to F is congruent to F as a set in E^3 (reflection admitted). This paper was written before the author established rigidity for closed convex surfaces without any differentiability hypothesis [see Doklady Akad. Nauk SSSR (N.S.) 79, 739–742 (1951); these Rev. 13, 271]. The methods of the latter paper permit one to reduce the differentiability assumption of the present paper. *H. Busemann* (Los Angeles, Calif.).

Aleksandrov, A. D. On triangles on convex surfaces. Doklady Akad. Nauk SSSR (N.S.) 50, 19–22 (1945). (Russian)

Aleksandrov, A. D. Curvature of convex surfaces. Doklady Akad. Nauk SSSR (N.S.) 50, 23–26 (1945). (Russian)

Aleksandrov, A. D. Convex surfaces as surfaces of positive curvature. Doklady Akad. Nauk SSSR (N.S.) 50, 27–30 (1945). (Russian)

Aleksandrov, A. D. An isoperimetric problem. Doklady Akad. Nauk SSSR (N.S.) 50, 31–34 (1945). (Russian)

All these results have since also appeared in the author's book, Intrinsic geometry of convex surfaces [Moscow-Leningrad, 1948; these Rev. 10, 619] and are mentioned in the review. *H. Busemann* (Los Angeles, Calif.).

Vietoris, L. Ein einfacher Beweis des Vierscheitelsatzes der ebenen Kurven. Arch. Math. 3, 304–306 (1952).

The author gives a simple proof of the theorem that every simple closed plane curve of class C'' has at least four vertices. The proof is based on a theorem of K. Zindler [Monatsh. Math. Phys. 31, 87–102 (1920), p. 88] assuring the existence of a minimum of the curvature interior to certain arcs which are tangent to a circle at two points. It might be noted that the reviewer established a somewhat stronger form of Zindler's theorem [Bull. Amer. Math. Soc. 50, 564–578 (1944), Lemma 4.1; these Rev. 6, 100] and then used substantially the method of the present paper to establish the same theorem. *S. B. Jackson* (College Park, Md.).

***Santaló, Luis A. Problems of integral geometry.** Symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Diciembre, 1951, pp. 23–40. Centro de Cooperación Científica de la Unesco para América Latina, Montevideo, Uruguay, 1952. (Spanish)

Algebraic Geometry

***Gauthier, L. Quelques travaux récents concernant la classification des courbes algébriques.** Deuxième Colloque de Géométrie Algébrique, Liège, 1952, pp. 29–39. Georges Thone, Liège; Masson & Cie, Paris, 1952. 375 Belgian francs; 2625 French francs.

This paper is concerned with the classification of algebraic curves of order n in two dimensions having three ordinary singular points of multiplicities a, b, c respectively with $a\geq b\geq c\geq 1$. Such curves are referred to as Drach Models, after M. J. Drach. The author shows that if p is the genus

of the curve, the following conditions hold:

$$(2n-3)^2-8p+2=(2a-1)^2+(2b-1)^2+(2c-1)^2,$$
$$n-2\leqq p\leqq \tfrac{1}{2}(n-1)(n-2),\quad a+b+c\leqq n.$$

Since the quantity on the left of the first equation is congruent to 3 (mod 8), positive integral values of a, b, c exist satisfying the equation. Using these results, the curves of genus less than 8 are listed. *B. W. Jones.*

Nagell, Trygve. Un théorème arithmétique sur les coniques. Ark. Mat. 2, 247–250 (1952).
Nagell, Trygve. Remarques sur les corps résolvants des coniques, cubiques et quartiques. Ark. Mat. 2, 379–384 (1952).

In the first of these two papers it is shown that, if a conic C belonging to a field Ω does not contain any point belonging to Ω, then the same occurs in any algebraic extension of Ω of odd order. Also a previous similar result concerning plane cubics is recalled [T. Nagell, Nova Acta Soc. Sci. Upsaliensis (4) 13, no. 3 (1942); these Rev. 8, 315], and another one on quartics is obtained. The argument of the proofs is completed in the second paper, where examples are also given and the result on quartics is modified by showing that, if a plane irreducible quartic belongs to Ω and contains a set (belonging to Ω) of an odd number of points, then the quartic contains a triplet of points belonging to Ω. [For extensions to algebraic curves of arbitrary order and genus, cf., B. Segre, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 13, 335–340 (1952).] *B. Segre* (Rome).

Beatty, S., and Lane, N. D. A symmetric proof of the Riemann-Roch theorem, and a new form of the unit theorem. Canadian J. Math. 4, 136–148 (1952).

The authors formulate and prove a Riemann-Roch theorem for a reducible algebraic curve (over the complex numbers) with no multiple component, and they prove that the form of their theorem is invariant with respect to changes of the independent variable, provided it does not remain constant on any of the component curves. The reviewer points out that it would be easy to derive directly from the classical Riemann-Roch theorem a similar theorem for direct sums of fields of algebraic functions of one variable which would have an invariant formulation.
C. Chevalley (New York, N. Y.).

Fusa, Carmelo. Sulla disuguaglianza di Noether. Boll. Un. Mat. Ital. (3) 7, 135–136 (1952).

Let $|C_n|$ be an irreducible linear system of plane algebraic curves of positive dimension, of order n, genus p, and grade d with multiplicities $h_0, h_1, \cdots, h_\nu$ at the $\nu+1\geqq 3$ base points so named that $h_0\geqq h_1\geqq\cdots\geqq h_\nu$. From well-known relations between these integers, the author deduces that $d<h_2(d-2p+2)$ implies that $h_0+h_1+h_2>n$.
G. B. Huff (Athens, Ga.).

Turri, Tullio. Quartiche di diramazione riducibili relative ad involuzioni di Geiser. Rend. Sem. Fac. Sci. Univ. Cagliari 21 (1951), 116–121 (1952).

Il ne peut exister de transformations involutives laissant fixe un réseau de cubiques avec six points-base, car la courbe de ses points unis devrait coïncider avec une courbe du réseau homaloïdal des quintiques avec ces six points doubles; la courbe des points unis d'une involution n'a pas de points multiples en dehors des points fondamentaux. Il en résulte qu'une involution conservant les cubiques par 7 points ou par 5 points ne peut avoir une courbe de points unis décomposée, car au premier cas le système des cubiques serait plus ample, au second, la conique fondamentale appartiendrait à cette courbe de points unis. Si l'involution est quadratique avec faisceau de droites unies, la conique des points unis se décompose en deux droites si deux points fondamentaux sont infiniment voisins, la quartique de diramation du plan double correspondant se décompose en deux coniques. Pour l'homologie harmonique la courbe des points unis est formée de l'axe d'homologie et du voisinage du centre, la quartique de diramation se décompose en une droite et une cubique rationnelle. *B. d'Orgeval* (Alger).

Turri, Tullio. Punti uniti in una trasformazione antibirazionale involutoria del piano. Rend. Sem. Fac. Sci. Univ. Cagliari 21 (1951), 126–130 (1952).

Une telle transformation étant le produit de celle par imaginaires conjugués et d'une involution réelle du second ordre, I_2 a ou non des points unis selon que l'involution a ou non des circuits réels finis ou infinitésimaux de points unis, un circuit infinitésimal étant formé du voisinage d'un point uni réel de I_2 où sont unies les directions qui en sont issues; lorsque ces points unis existent, ils forment du point de vue réel un ensemble ∞^2; démonstration par étude successive des types d'involution réelle. *B. d'Orgeval* (Alger).

Turri, Tullio. Rappresentazione piana di involuzioni sopra superficie di Eckardt. Rend. Sem. Fac. Sci. Univ. Cagliari 21 (1951), 122–125 (1952).

Soit F une surface cubique conservée par une homologie harmonique de centre au point d'Eckardt où le plan tangent coupe F selon trois droites e_i. On a sur F une involution Ω; si on considère alors deux droites c et d de F rencontrant une e_i les droites s'appuyant à c et d et coupant ultérieurement F en des points homologues de Ω sont en involution; sur le plan d'homologie, ces droites coupent une involution de de Jonquières à cubique de points unis.
B. d'Orgeval (Alger).

Haradze, A. On a class of algebraic surfaces. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 15, 95–99 (1947). (Georgian. Russian summary)

The author considers surfaces Σ_n defined by the parametric equations

$$x=A_n(u,v)=\tfrac{1}{3}[(1+u+v)^n+(1+\omega u+\omega^2 v)^n+(1+\omega^2 u+\omega v)^n],$$
$$y=B_n(u,v)=\tfrac{1}{3}[(1+u+v)^n+\omega(1+\omega u+\omega^2 v)^n+\omega^2(1+\omega^2 u+\omega v)^n],$$
$$z=C_n(u,v)=\tfrac{1}{3}[(1+u+v)^n+\omega^2(1+\omega u+\omega^2 v)^n+\omega(1+\omega^2 u+\omega v)^n].$$

From the author's summary.

Godeaux, Lucien. Mémoire sur les surfaces multiples. Acad. Roy. Belgique Cl. Sci. Mém. Coll. in 8° (2) 27, no. 1626, 80 pp. (1952).

Let F be an algebraic surface having a birational self-transformation T, cyclic of prime order p, without fundamental points, and with only a finite number of fixed points, this generating an involution I_p on F. A linear system $|C_0|$ is constructed, compounded with I_p and free from base points, so that its projective model Φ is an image of I_p. The problem of this monograph is the behaviour of curves of $|C_0|$ made to pass a certain number of times through a fixed point A of T (united point of I_p) and the nature of the corresponding point A' of Φ.

A is of the first kind (espèce) if every point of its first neighbourhood is united for I_p, i.e., if every branch through A is transformed by T into a branch with the same tangent. In this case the curves of $|C_0|$ passing through A have a p-ple point there with independently variable tangents; A' is a p-ple point of Φ, the tangent cone being irreducible, rational, and normal.

A is of the second kind if its first neighbourhood contains an involution belonging to I_p with united points A_1, A_α. In this case the transformation by T of the tangent plane at A is of the form

$$x_0':x_1':x_\alpha'=x_0:\xi x_1:\xi^\alpha x_\alpha=x_0:\eta^\beta x_1:\eta x_\alpha$$

where $\alpha\beta\equiv1 \pmod p$, and $\xi=\eta^\beta$, $\eta=\xi^\alpha$ are pth roots of unity; $x_1=0$, $x_\alpha=0$ are the tangents AA_α, AA_1. There is then a sequence of linear systems,

$$|C_0'|, |C_0''|, \cdots, |C_0^{(\nu)}|, |C_0^{(\nu+1)}|$$

where $\nu=\frac{1}{2}(p-1)$, contained in $|C_0|$, each obtained by imposing one further condition on curves of the previous system, $|C_0'|$ being that of all curves of $|C_0|$ which pass through A. $|C_0^{(i)}|$ $(i\leqq\nu)$ has a $(\lambda_i+\mu_i)$-ple point at A, λ_i tangents coinciding with AA_α and μ_i with AA_1, where λ_i, μ_i are solutions of the congruences

$$\lambda+\alpha\mu\equiv0,\quad \beta\lambda+\mu\equiv0 \pmod p$$

satisfying $\lambda_1+\mu_1<\lambda_2+\mu_2<\cdots<\lambda_\nu+\mu_\nu\leqq p$. On the other hand, $|C_0^{(\nu+1)}|$ has a p-ple point at A with variable tangents. There are also systems $|C_1|$, $|C_\alpha|$ compounded with I_p and each having p consecutive simple base points beginning with AA_1, AA_α, respectively; and the general curve of each of these has with the general curve of $|C_0^{(i)}|$ p intersections absorbed in A and its neighbourhoods.

The study of the point A' of Φ involves a further classification of united points of the second kind; in the equations

$$\lambda_1+\alpha\mu_1=h_\alpha p,\quad \beta\lambda_1+\mu_1=h_\beta p$$

according as both, one, or neither of the coefficients h_α, h_β have the value 1, A is said to be of the first, second, or third category.

If A is of the first category, the general curve of $|C_0'|$ has λ_1 branches through the first β base points of C_α and μ_1 through the first α base points of C_1; the last points of these sequences represent the two constituents of the first neighbourhood of A', which is a $(\lambda_1+\mu_1)$-ple point of Φ, the tangent cone breaking up into two constituents, each normal and rational, of orders λ_1, μ_1 respectively, with one common generator. Consecutive to A' along this common generator there is in general a double point, of type B_{t+1}, i.e., of the kind whose partial neighbourhoods are t rational curves of grade -2, intersecting consecutively; according as t is even or odd this consists of $\frac{1}{2}t$ consecutive binodes of which the last is ordinary, or of $\frac{1}{2}(t-1)$ binodes followed by a conic node; here t is given by

$$t+1=\frac{\alpha-1}{\lambda_1}=\frac{\beta-1}{\mu_1}=\frac{p-\lambda_1-\mu_1}{\lambda_1\mu_1}.$$

If A is of the second category the tangent cone at A' has three constituents, and if it is of the third category, four constituents; these are rational and normal, and can be arranged in sequence so that consecutive cones have a common generator, on which is generally a binode consecutive to A; the arithmetic giving the orders of these cones and the suffixes of the binodes is hard to follow and too complicated to summarise.

A final section gives some relations between the invariants of F and Φ. *P. Du Val* (Bristol).

Godeaux, Lucien. Sur quelques points de diramation de seconde espèce et de troisième catégorie d'une surface multiple. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 38, 755–765 (1952).

This is an exercise in detail on a particular united point of the second kind and third category [see the preceding review, whose terminology and notation is assumed throughout]. The values $p=37$, $\alpha=29$ are taken, from which it follows that $\beta=23$, $\lambda_1=3$, $\mu_1=5$, so that

$$\lambda_1+\alpha\mu_1=4p,\quad \beta\lambda_1+\mu_1=2p$$

and the point is in fact of the third category. A' is a quadruple point of Φ, the tangent cone consisting of four planes (σ_1), (τ_1), (τ_2), (σ_2), consecutive pairs having a line in common; in the common direction of (τ_2), (σ_2), Φ has an ordinary binode (type B_3) consecutive to A', which has no other singular point in its neighbourhood. The partial neighbourhoods of A' are six curves σ_1, τ_1, τ_2, ρ_1, ρ_2, σ_2 of virtual grades $-2, -3, -3, -2, -2, -2$, intersecting consecutively, of which ρ_1, ρ_2 represent the partial neighbourhoods of the binode. The linear systems $|C_0'|, \cdots, |C_0^{(18)}|$ are studied in detail, their base multiplicities and the proximity relations of their base points being given, as also the corresponding systems $|\Gamma_0'|, \cdots, |\Gamma_0^{(18)}|$ of hyperplane sections of Φ. *P. Du Val* (Bristol).

Godeaux, Lucien. Sur un point de diramation d'une surface multiple en lequel le cone tangent se décompose en quatre parties. Univ. e Politecnico Torino. Rend. Sem. Mat. 11, 203–222 (1952).

Godeaux, Lucien. Une généralisation des surfaces desmiques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 38, 892–897 (1952).

Construction d'un faisceau de surfaces algébriques contenant trois surfaces dégénérées en quatre parties. Examen d'un cas particulier où les surfaces de ce faisceau sont irrégulières. *Author's summary.*

Conforto, Fabio. Introduzione elementare alla geometria simplettica. Univ. e Politecnico Torino. Rend. Sem. Mat. 11, 93–109 (1952).

Expository paper.

Du Val, Patrick. Algebraic loci whose curve sections are hyperelliptic. J. London Math. Soc. 28, 1–8 (1953).

L'auteur s'occupe ici des variétés algébriques "$^rH_k^n$", d'ordre n et dimension $k\geqq2$ dans un espace à r dimensions, dont les courbes sections par des espaces à $r-k+1$ dimensions sont hyperelliptiques de genre π $(\geqq2)$. Il se pose la question de trouver les conditions auxquelles doivent satisfaire les nombres n, k, π afin que $^rH_k^n$ ne soit pas un cône. Après quelques remarques générales sur les variétés $^rH_k^n$, et apres avoir démontré qu'elles peuvent être obtenues comme sections d'une R^m_{k+1} rationnelle lieu de ∞^1 espaces S_k avec des quadriques (contenant éventuellement un certain nombre de S_k générateurs de R^m_{k+1}), il se sert d'une représentation paramétrique de R^m_{k+1} et de l'équation de $^rH_k^n$ sur R^m_{k+1}. Il trouve que, lorsque $\pi+2\leqq n\leqq2\pi+2$, la variété $^rH_k^n$ existe sans être nécessairement un cône sans aucune condition pour k. Lorsque $n=2\pi+3$, on doit avoir $k\leqq2\pi+3$; et pour $n\geqq2\pi+4$ on doit avoir $n+2k\leqq4\pi+8$.

On peut déduire d'ici la condition suivante pour tous les cas: $n \leq 2k(\pi+1)/(k-1)$. *E. G. Togliatti* (Gênes).

★Néron, A. La théorie de la base pour les diviseurs sur les variétés algébriques. Deuxième Colloque de Géométrie Algébrique, Liège, 1952, pp. 119–126. Georges Thone, Liège; Masson & Cie, Paris, 1952. 375 Belgian francs; 2625 French francs.

Let V be an algebraic variety, G the group of divisors on V, and G_a the group of divisors algebraically equivalent to 0. Then the theorem of Severi states that G/G_a is a finitely generated group. This theorem has been established by Severi in the case where the basic field is that of complex numbers (by transcendental methods). The author briefly sketches a proof of the extension of this theorem to an arbitrary basic field. The proof proceeds by induction on the dimension n of V, and each step of the induction involves an argument of "infinite descent" of the type of the one which was used by A. Weil in the proof of the corresponding theorem for curves over a field of algebraic numbers.

The problem is first reduced to the case where V is fibered by a family of curves; the parameter variety of this family, say $\mathfrak{M}$, is then of dimension $n-1$, and it is to $\mathfrak{M}$ that the inductive assumption is applied. Let H_0 be the group of divisors in G which cut out a divisor of degree 0 on the generic fiber of V, H the group of divisors whose projections on $\mathfrak{M}$ are of dimensions $\leq n-2$ and $H_a = G_a + H$. Let H' be the group of reciprocal images of divisors on $\mathfrak{M}$ under the projection of V on $\mathfrak{M}$; then $(G_a+H')/G_a$ is isomorphic to a factor group of $G(\mathfrak{M})/G_a(\mathfrak{M})$ and is therefore finitely generated by the inductive assumption. If $H_a = H + G_a$, then $H_a/(G_a+H')$ is easily seen to be finitely generated. The problem is therefore reduced to proving that H_0/H_a is finitely generated (G/H_0 being a cyclic group).

In order to do this, let $C(M)$ be the generic fiber of V (M generic on $\mathfrak{M}$), $J(M)$ its Jacobian variety, and $\mathfrak{J}$ the locus of $J(M)$ over the field of definition k of V. To every divisor in H_0 there is associated a section of the fibered variety $\mathfrak{J}$, and these sections form a group h isomorphic to H_0/H_l, where H_l is generated by H and by the group G_l of divisors linearly equivalent to 0 on V. It is first proved that, for any $s>0$, the group h/sh is finite. The proof is based on properties of abelian varieties. This paves the way for an application of the method of infinite descent. Let $Z_1, \cdots, Z_r$ be representatives for the classes of h modulo sh. To any Z in h we may associate an infinite sequence $Z^{(i)}$ such that $Z^{(i-1)} = sZ^{(i)} + Z_{l(i)}$, $1 \leq l(i) \leq r$. Let h_a be the image of H_a/H_l in h. Then the problem is reduced to showing that $Z^{(i)}$ ultimately belongs to some finite set of cosets of h modulo h_a which does not depend on Z. This is accomplished by making use of a projective model of $\mathfrak{J}$ and showing that $Z^{(i)}$ is represented on this model by a cycle of bounded degree.

As an application, the author derives a generalization of A. Weil's theorem on curves to the case of a curve defined over an arbitrary field of finite type (i.e., absolutely generated by a finite number of elements). *C. Chevalley.*

Igusa, Jun-ichi. Normal point and tangent cone of an algebraic variety. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 27, 189–201 (1952).

Let V^d be a projective variety, P a point of V. Denote $H_i(V)$ and $H_i(V, P)$ the i-dimensional global holotomy group of V and the i-dimensional local holotomy group of V at P as defined by the reviewer [J. Math. Pures Appl. (9) 30, 159–205, 207–274 (1951); C. R. Acad. Sci. Paris 228, 158–159, 292–294 (1949); these Rev. 13, 980; 10, 732; 13, 1138]. If the representative cone W^{d+1} of V is normal at its vertex O, then $H_d(W, O)$ is isomorphic to $H_{d-1}(V)$, which is itself the direct sum of the Picard variety of V and of a finitely generated abelian group. In the classical case, and when V is non-singular, this holotomy group is the direct sum of the compact dual group of the one-dimensional homology group of W at P and of a free abelian group; this is proved by studying the characteristic class of $W-O$ considered as a fibre bundle over V.

We now come back to the abstract case. The associated graded ring $\bar{R}$ of the local ring R of V^d at P is the coordinate ring of a conic algebraic set $\bar{V}$; the proof that $\bar{V}$ is the tangent cone of V at P (i.e., the set of lines D through P such that $i(P; V\cdot D) < m(P, V)$) is sketched in an appendix. The relations between a local ring R and its associated graded ring $\bar{R}$ (which is also the associated graded ring of the completion R^* of R) show that, if $\bar{V}$ is a variety, then V is analytically irreducible at P, and that, if $\bar{V}$ is normal at P, then V is also normal at P and so is the unique algebroid sheet V^* of V at P. When $\bar{V}$ is normal at P, the fact that the holotomy groups $H_{d-1}(V, P)$, $H_{d-1}(V^*, P)$, $H_{d-1}(\bar{V}, P)$ are isomorphic to the ideal class groups of R, R^*, $\bar{R}$ allows the definition of a canonical isomorphism of $H_{d-1}(V, P)$ into $H_{d-1}(V^*, P)$, and of a canonical homomorphism of $H_{d-1}(V^*, P)$ into $H_{d-1}(\bar{V}, P)$. *P. Samuel.*

Chow, Wei-Liang. On the quotient variety of an Abelian variety. Proc. Nat. Acad. Sci. U. S. A. 38, 1039–1044 (1952).

Let A be an Abelian variety of dimension r defined over a field K. By an algebraic subgroup X of A is meant a subgroup which is also a bunch of subvarieties. Assume that X is of dimension r and normally algebraic over K. Then the author proves that there exist an Abelian variety $A(X; K)$ defined over K and a homomorphism F, defined over K, of A onto $A(X; K)$ such that the kernel of F is X. Moreover, if H is any homomorphism defined over K of A into an Abelian variety B defined over K, and if the kernel of H is X, then H factors into F and a $1-1$ homomorphism of $A(X; K)$ into B defined over K. The author first considers the case where X is separable over K. In that case, the method of proof is roughly as follows. Let Z be the r-cycle on A, sum of the varieties of the bunch X each counted with multiplicity 1. Then the points of $A(X; K)$ are the cycles $T_x(Z)$ translations of Z by the elements x of the group A. The algebraic structure of $A(X; K)$ is the one induced by the algebraic structure of the set of all r-cycles of a given degree d in the ambient projective space; F is the mapping $x \rightarrow T_x(Z)$. In order to extend the result to the case where X is not separable over K, the author proceeds as follows. Let $\mathfrak{L}$ be the set of all subfields of $K(x)$ (where x is a generic point of A over K) which are fields of rational functions of Abelian varieties images of A under homomorphisms of kernel X; this set is not empty by the result established above. It is shown that $\mathfrak{L}$ has a maximal element L, and L is the field of rational functions on the desired variety $A(X; K)$. *C. Chevalley* (New York, N. Y.).

Kawai, Ryoichiro. A note on the Riemann-Roch-Weil's theorem. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 27, 123–131 (1952).

Suppose that K/k is a function field of one variable over the algebraically closed coefficient field k of characteristic $p \neq 0$. The author shows, using the fact that each completion

of K with respect to a valuation, which is trivial on k, admits for each integer n prime to p essentially one cyclic extension of degree n, that A. Weil's generalization of the Riemann-Roch theorem to rings of matrices over K can be extended, provided the prescribed signatures are relatively prime to p. He first establishes a modified form of Witt's generalization of the Riemann-Roch theorem [Math. Ann. **110**, 12–28 (1934)] by introducing Weil's invariance conditions for the given signatures and following in essence Weil's method of counting multiplicities and constants [A. Weil, J. Math. Pures Appl. (9) **17**, 47–87 (1938)]. Weil's theorem then follows quickly by applying the above generalization to a suitable divisor. *O. F. G. Schilling* (Chicago, Ill.).

Differential Geometry

Gaddum, Jerry W. Metric methods in integral and differential geometry. Amer. J. Math. **75**, 30–42 (1953).

In Part I of the paper are given a few simple propositions on the "spread" of a metric arc A, defined by W. A. Wilson [same J. **57**, 62–68 (1935)]. The principal contribution in Part II consists in a metric proof of the first fundamental theorem of curve theory in euclidean E_3, formulated as follows: If two arcs A and A^* of E_3 possess at each point Menger curvatures R, R' and Blumenthal torsions T, T' respectively and if A and A^* can be put into one-to-one correspondence $p^*=f(p)$ in such a way that the arc lengths between pairs of corresponding points are equal, then the arcs are congruent. Pattern of the proof: Two regular n-polygons (Blumenthal's n-lattices) $p_0, p_1, \cdots, p_n$ and $p_0', p_1', \cdots, p_n'$ are inscribed in A and A^* respectively; for n sufficiently large p_i' is arbitrarily close to $f(p_i)$ and the correspondence $p_i \leftrightarrow p_i'$ is arbitrarily close to a congruence. The Frenet-Serret formulas are established for any arc in E_3 possessing Menger curvature and Blumenthal torsion. [Remarks by the reviewer: (1) Wilson's spread is a paratingent in the reviewer's sense [Acad. Roy. Belgique. Bull. Cl. Sci. (5) **22**, 968–984 (1936); C. R. Acad. Sci. Paris **206**, 1242–1244 (1938)] for $\theta(x, y)=f(x)f(y)/xy$. Theorems I.2.1. and I.2.2. are true for "abstract" paratingents. (2) Regarding $f(x)f(y)$ as a function of the interval (x, y) and $s(\bar{a}, b)$ as its Burkill integral, Part I from Theorem I.2.2. onwards is partially contained in Pauc, La méthode métrique en calcul des variations, Hermann, Paris, 1941, pp. 24–25 [these Rev. **7**, 67]. (3) There is a metric study of torsion in Alexits, Compositio Math. **6**, 471–477 (1939). (4) The existence of a "strong" binormal follows obviously from the existence of a "strong" osculating plane. (5) A comparative study of Alt and classic curvatures implying Theorem II.5.1. as a corollary is to be found in van der Waag's Amsterdam Thesis [Nederl. Akad. Wetensch. Proc. Ser. A. **55**, 92–110, 275–303 (1952), §5.3; these Rev. **14**, 83]. (6) Not proved is the assertion p. 39 that in the vector equation $x=x(s)$ of a "bi-regular" arc the functions x_i are three times differentiable. Its validity would mean that the existence everywhere of the Menger curvature and the Blumenthal torsion T for an arc in E_3 should imply the existence of dR/ds. It seems that the following example invalidates this assertion: $x_1=e^t\cos t$, $x_2=e^t\sin t$, $x_3=e^t-1$ when $0\leq t\leq \pi/2$; $x_1=\cos t+\sin t$, $x_2=\cos t-\sin t-1$, $x_3=t$ when $-\pi/2\leq t\leq 0$. The arc so defined possesses everywhere continuous classic curvature and torsion, and yet the functions $x_i(t)$ are not three times differentiable for $t=0$.] *Chr. Pauc.*

Leibin, A. S. Spherical mapping of Boy's surface. Učenye Zapiski Har'kov. Gos. Univ. **28**, Zapiski Naučno-Issled. Inst. Mat. Meh. i Har'kov. Mat. Obšč. (4) **20**, 127–154 (1950). (Russian)

The spherical mapping of Boy's surface (a surface in E^3 with continuous Gauss curvature realizing topologically the projective plane) is discussed in detail. The work is not without interest; it is actually suggested in Hilbert and Cohn-Vossen's Anschauliche Geometrie [Springer, Berlin, 1932, p. 283], but it does not lend itself to a brief description. *H. Busemann* (Los Angeles, Calif.).

***Bukreev, B. Ya. Planimetriya Lobačevskogo v analitičeskom izloženii. [Lobačevskii's planimetry in an analytic exposition.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 127 pp. 4.25 rubles.

The book develops hyperbolic geometry from the point of view of differential geometry. It is very elementary and reliable, but without zest and elegance. The line element is usually taken in the form $ds^2=R^2y^{-2}(dx^2+dy^2)$ or the related form $ds^2=du^2+e^{-2u/R}dv^2$. The content of the nine chapters is as follows: I. The geodesics as solutions of the Euler equation, Gauss curvature, geodesic curvature; II. Formula for the parallel angle; III. Equidistant curves and limit circles; IV. Motions; V. Geodesic triangles, concurrence of altitudes, medians, etc.; VI. Hyperbolic trigonometry; VII. Lambert quadrilaterals; VIII. Area of triangles and circles, length of circles; IX. Realization as geometry on the pseudosphere. *H. Busemann* (Los Angeles, Calif.).

Paquet, Henriette. Sur certains couples de surfaces. Bull. Soc. Roy. Sci. Liège **21**, 364–368 (1952).

Nous nous proposons d'étudier des couples de surfaces qui se correspondent avec conservation des asymptotiques et de telle sorte que la congruence formée par les droites joignant les points homologues soit une gerbe de rayons.

Author's summary.

Nikoladze, G. On continuous systems of geometric figures. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **15**, 27–93 (1947). (Georgian)

A translation of Nikoladze's thesis, "Sur les systèmes continus de figures géométriques" [Blanchard, Paris, 1928].

Krishna, Shri. On the congruence of Ribaucour. Ganita **3**, 59–70 (1952).

L'auteur reprend l'étude des congruences rectilignes de Ribaucour, définies par deux surfaces (une surface directrice et une surface de référence) se correspondant avec orthogonalité des éléments linéaires. Il exprime les premiers tenseurs des deux surfaces directrice et de référence d'une congruence de Ribaucour, ainsi que certaines quantités $e_{\alpha\beta}$ attachées à ces surfaces, au moyen des coefficients des formes quadratiques de Kummer et de Sannia de la congruence, et fait différentes applications des résultats obtenus. Les cas où la congruence est normale ou formée par les normales à la surface de référence sont spécialement envisagés, et, dans le dernier cas mentionné, diverses relations entre courbes remarquables (minima, asymptotiques, de courbure ou de torsion géodésique) des deux surfaces directrice et de référence sont mises en évidence. *P. Vincensini.*

Mishra, R. S. Congruence of curves through points of a hypersurface. Ganita **3**, 37–40 (1952).

L'auteur envisage une hypersurface V_n d'un espace riemannien V_{n+1}, et une congruence de courbes de V_{n+1}

telle que par chaque point de V_n il passe une courbe de la congruence. Il considère les quantités $v_j = a_{\alpha\beta}N^\alpha\lambda^\beta{}_{;j}$, où $a_{\alpha\beta}$ est le tenseur fondamental de V_{n+1}, N^α la normale unitaire en un point quelconque de V_n et λ^β le vecteur unitaire tangent à la courbe de la congruence issue de ce point, et il donne, pour ces quantités, des expressions généralisant les expressions analogues relatives à l'espace euclidien. Si λ^α est normal à V_n on retrouve les équations généralisées de Mainardi-Codazzi pour une hypersurface de l'espace riemannien. *P. Vincensini* (Marseille).

Mishra, Ratan Shanker. Some properties of normal rectilinear congruences. Proc. Nat. Acad. Sci. India. Sect. A. **15**, 125–127 (1946).

Cette note donne une démonstration de la propriété bien connue des surfaces minima, d'avoir pour représentation sphérique de ses lignes de courbure un réseau orthogonal isotherme. *P. Vincensini* (Marseille).

Grove, V. G. The quadric of Lie. Proc. Amer. Math. Soc. **3**, 573–579 (1952).

Let (x^1, x^2, x^3, x^4) denote the coordinates of a generic point x of a surface S so normalized that they satisfy the differential equations $x_{uu} = \theta_u x_u + \beta x_v + px$, $x_{vv} = \theta_v x_v + \gamma x_u + qx$, $\theta = \log R$. Let a line l_2 in the tangent plane to S at x be determined by the points r, s defined by $r = x_u - bx, s = x_v - ax$. Let l_1 denote the reciprocal of l_2 with respect to S at x. Any point z on l_1 except x has coordinates of the form $z = x_{uv} - ax_u - bx_v + \varphi x$. As x generates the curve C through x of the one-parameter family of curves defined by $\mu du - \lambda dv = 0$, the point z describes a curve. The tangent of this curve intersects the tangent plane to S at x in a point whose projection from x on the line l_2 is denoted by $\bar{y}$. Let y denote the intersection of the tangent to C at x with the line l_2. The points y and $\bar{y}$ are corresponding points in a projectivity. This projectivity is a particular member of the family $F(\varphi)$ of such projectivities. In $F(\varphi)$ there is one involution. The particular point z which corresponds to this involution is called the involutory point I_1 on l_1. A point h of l_1, called the harmonic point on l_1, is defined as follows: The common intersector l_1 of the tangents to the loci of the points r and s which passes through x intersects these respective tangents in the points z_1, z_2. The point h is the harmonic conjugate of x with respect to the points z_1, z_2. Let g be the harmonic conjugate of x with respect to the harmonic point h, and the involutory point I_1. The locus of the point g as l_1 varies in the tangent plane is the quadric of Lie of S at x. The lines (gr) and (gs) are generators of this quadric, and hence the quadric of Lie is enveloped by the plane of g and l_2. The double points of the projectivities of the family $F(\varphi)$ are identical for all members of the family; and they are the intersections of the tangents to the Γ_1-curves of the congruence Γ_1 of lines l_1 with l_2. The above described constructions are also dualized in the paper. Some of the notions developed are used in modified settings to describe methods of generating other quadrics, notably those of Lane [Amer. J. Math. **48**, 204–214 (1926)] and Wilczynski [Trans. Amer. Math. Soc. **9**, 79–120 (1908)]. A fundamental concept employed in the method of generating these quadrics is the author's extension of the R_λ-associate of a line l_2 and the R_λ-derived line and curves developed by the reviewer [Trans. Amer. Math. Soc. **46**, 389–409 (1939); these Rev. **1**, 85]. The extension, called the $R_{\lambda\mu}$-associate of l_2 is defined as follows: Denote by C_λ, C_μ integral curves of the differential equations $dv - \lambda du = 0$, $du - \mu dv = 0$, respectively. As x generates the curve C_λ the point r describes a curve. As x moves along C_μ the point s describes a curve. The common intersector through x of the tangents to these loci of r and s is a line whose reciprocal with respect to S at x is called the $R_{\lambda\mu}$-associate of l_2. The $R_{\lambda\mu}$-derived line is defined to be the line joining x to the point of intersection of the $R_{\lambda\mu}$-associate of l_2 with l_2. *P. O. Bell.*

Lenz, Hanfried. Über Orthogonaltrajektorien. Math. Z. **57**, 46–64 (1952).

The orthogonal trajectories of a one-parameter family of surfaces furnish a mapping of the surfaces on one another. For general families the following theorem is established: The mappings of a family of surfaces on each other by their orthogonal trajectories are (a) area-preserving, (b) conformal, or (c) isometric if and only if the surfaces of the family are (a) minimal surfaces, (b) spheres or planes, (c) planes. For the case of families of spheres, since the mappings are conformal they are also cyclic, i.e., take circles into circles, and thus can be expressed as linear fractional transformations of the complex number sphere. Special attention is devoted to families of spheres which admit orthogonal trajectories that are spherical curves, and it is established that if a family of spheres admits two spherical orthogonal trajectories on distinct spheres, then all orthogonal trajectories of the family are spherical curves.

The question of closed orthogonal trajectories of families of spheres is treated. It is proved that in general a closed family of spheres has exactly two closed orthogonal trajectories, and that all other orthogonal trajectories approach these asymptotically by repeated running through of the family of surfaces. An extension of this result to families of ovaloids and families of m-dimensional spheres is given. Reference is made in the paper to a considerable amount of related work, including particularly work by Voss, Darboux, Bianchi, and Löbell. *S. B. Jackson.*

Beurling, Arne. Sur la géométrie métrique des surfaces à courbure totale $\leqq 0$. Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 7–11 (1952).

Let R be a domain of the complex z-plane and let there be defined in R a differential metric $ds = \lambda(z)|dz|$ $(\lambda = e^u)$. Let R^∞ be the universal covering surface of R and let a, b be points of R^∞. The geodesic distance between a and b, denoted by L, is inf $\int_\Gamma \lambda(z)|dz|$ where Γ denotes the class of rectifiable paths in R^∞ joining a and b. If $\Gamma_1, \Gamma_2 \in \Gamma$, the distance D between Γ_1 and Γ_2 is defined by the Hausdorff metric. That is, D is the smallest number such that for every $\epsilon > 0$, Γ_1 (Γ_2) lies in the $(D+\epsilon)$-neighborhood of Γ_2 (Γ_1). By an ingenious and simple method, the following basic theorem is proved: If the Gaussian (total) curvature is non-positive (i.e., $u = \log \lambda$ is subharmonic) and $L_i = \int_{\Gamma_i} \lambda(z)|dz|$, $i = 1, 2$, then $D^2 + L^2 \leqq \frac{1}{2}(L_1^2 + L_2^2)$. It follows that if $\{\Gamma_n\}$ is a minimal sequence in Γ (i.e., $\int_{\Gamma_n} \lambda(z)|dz| \to L$), then $\{\Gamma_n\}$ is a Cauchy sequence in terms of the D-distance. In terms of the original domain R, this enables one to prove the existence of minimizing curves within homotopy classes.

G. A. Hedlund (New Haven, Conn.).

Hartman, Philip, and Wintner, Aurel. Envelopes and discriminant curves. Amer. J. Math. **75**, 142–158 (1953).

If a surface S in Euclidean 3-space is of class C^n $(n \geqq 2)$, the mean curvature H and the Gaussian curvature K are defined and are of class C^{n-2}. This paper considers the possible converse theorems. Standard theorems in the theory of elliptic differential equations give two partial converse

theorems. (1) If S is of class C^n ($n\geqq 2$) and H is of class C^n, then S has a parametric representation of class C^{n+1}. (2) If S is of class C^n ($n>2$) and K is of class C^n and $K>0$, then S has a parametric representation of class C^{n+1}. Refinements on these theorems are the following. (3) If S is of class C^n and (α) $n>2$, (β) K is of class C^{n-1}, (γ) H is of class C^{n-1}, (δ) $H^2>K$, then S has a parametric representation of class C^{n+1}. The case $n=2$ is undecided. (4) The statement (3) is true if (α) and (β) are replaced by: (α^*) $n\geqq 2$, (β^*) K is a constant. (5) The statement (3) is false if any one of the assumptions (β), (γ), (δ) is omitted (whether $n>2$ or $n=2$).
C. B. Allendoerfer (Seattle, Wash.).

Hartman, Philip, and Wintner, Aurel. On the curvatures of a surface. Amer. J. Math. **75**, 127–141 (1953).

Cauchy has considered conditions for the existence of an envelope of the solution curves of the real differential equation: (1) $y'=f(x, y)$. A modern, rigorous statement of these results is proved as follows: Let $f(x, y)$ be a continuous function on the rectangle R: $-a\leqq x\leqq a$, $0\leqq y\leqq b$, such that $f(x, 0)\equiv 0$, and such that there exists a continuous function $\phi(y)$ satisfying $\phi(y)>0$ if $y>0$, $\phi(0)=0$, $\int_0^b dy/\phi(y)<\infty$, for which $f(x, y)>\phi(y)$. Put $M=\max_R f(x, y)$ and let D be the (x, c)-set D: $c\leqq x\leqq \min(a, c+b/M)$, $-a<c<a$. Then there exists in D a function $y=y(x; c)$ which is continuous from the left with respect to c (uniformly in x) and which has the properties: $y(x; c)$ is a solution of (1) (for fixed c) and $y(x; c)\geqq 0$ holds according as $x\gtreqless c$. Finally (a portion of) the x-axis is an envelope of this one-parameter family of solutions.

This theorem is then applied to the classical problem of nets satisfying: $a(x, y)dx^2+2b(x, y)dxdy+c(x, y)dy^2=0$, where the discriminant $ac-b^2$ plays a leading role. In particular, the envelopes of a family of asymptotic curves are discussed. *C. B. Allendoerfer* (Seattle, Wash.).

Wong, Yung-Chow. A new curvature theory for surfaces in a Euclidean 4-space. Comment. Math. Helv. **26**, 152–170 (1952).

The known theory of the local differential geometry of a two-dimensional surface (A) imbedded in a Euclidean R_4 depends upon two conics lying in the normal plane. The curvature ellipse (G) is the locus of the end points of the normal curvature vectors corresponding to the various tangent vectors; the Kommerell conic (K) is the locus of the intersections of the normal plane with normal planes at "consecutive" points. In terms of these the concepts of minimal point, axial point, etc. are defined.

In this paper the author considers the two angles $d\psi_1$ and $d\psi_2$ between the tangent plane at a point A and a tangent plane at a consecutive point A^*. Let the distance AA^* be ds. Then $d\psi_1/ds$ and $d\psi_2/ds$ are defined as the two first curvatures of (A) at the point A in the direction AA^*. It is proved that a necessary and sufficient condition for A to be minimal or axial is that these curvatures be independent of the direction at A. Also a non-minimal surface is completely determined to within a motion by its linear element and its two first curvatures.

In the course of the paper numerous results are proved concerning the relationship between the angle directions in the tangent plane and special properties of the ellipse G.
C. B. Allendoerfer (Seattle, Wash.).

Hopf, H., und Voss, K. Ein Satz aus der Flächentheorie im Grossen. Arch. Math. **3**, 187–192 (1952).

Following a suggestion of Pólya the authors prove two theorems, the second being a consequence of the first. These theorems give an easy proof of the known result that the only convex surface of constant mean curvature is the sphere. I. Let F and $\bar{F}$ be oriented closed surfaces, and let there be a correspondence $p\leftrightarrow\bar{p}$ such that (a) orientation is preserved, (b) the segments $p\bar{p}$ are all parallel, (c) the mean curvatures at p and $\bar{p}$ are equal, (d) the surfaces have no cylindrical elements whose generators are parallel to $p\bar{p}$. Then F can be transformed into $\bar{F}$ by a translation. II. A closed surface F is called convex in a direction if every line in this direction either does not intersect F, is tangent at exactly one point, or intersects F in exactly two points. If there is a direction in which F is convex for which the mean curvatures at every pair of points of intersection are equal, then F has a plane of symmetry perpendicular to this direction.

Similar results are proved for surfaces with boundaries, and for curves in the plane. *C. B. Allendoerfer*.

Shouten, I. A. [Schouten, J. A.], and Van-der-Kul'k, V. [van der Kulk, W.] On Pfaff's problem and its generalizations. Trudy Sem. Vektor. Tenzor. Analizu **6**, 249–256 (1948). (Russian)

An expository account of the authors' researches on Pfaff's problem; these were published later in their book, "Pfaff's problem and its generalizations", Oxford, 1949 [these Rev. **11**, 179].

Lotze, A. Zur vektoriellen Deutung Pfaff'scher Formen und der mit ihnen verbundenen Operationen in der Differentialgeometrie. Jber. Deutsch. Math. Verein. **56**, Abt. 1, 21–26 (1952).

This is an exposition of the ideas of the curl and divergence of a vector in a Riemannian space. Vector notation and Grassmann algebra are used in conjunction with the algebra of Pfaffian forms. *C. B. Allendoerfer* (Seattle, Wash.).

Castoldi, Luigi. Estensione a tensori qualunque negli spazi di Riemann di alcuni teoremi fondamentali dell'analisi vettoriale. Rend. Soc. Ital. Sci. Accad. dei XL (3) **27**, 245–253 (1949).

Having defined the inner and outer products of two tensors, the supplementary vector of a tensor is defined as the inner product of the tensor with the completely skew-symmetric tensor of maximum order, a vector being generalized to mean any skew-symmetric tensor. The vector product of two tensors is the supplementary vector of their outer product and the vector of a tensor is the supplemental vector of the supplemental vector. The gradient and curl of a tensor are defined as the vector of the covariant derivative and the supplementary vector of the gradient respectively, and some integral theorems analogous to Stokes' theorem are obtained. *A. J. McConnell* (Dublin).

Bompiani, Enrico. Proprietà d'immersione di una varietà in uno spazio di Riemann. Rend. Sem. Mat. Fis. Milano **22** (1951), 1–24 (1952).

A survey of the embedding theory of a manifold in a Riemannian space. It is shown that the embedding (or Euler's) tensor, introduced by the author in 1921 [Ist. Veneto Sci. Lett. Arti. Atti Cl. Sci. Mat. Nat. (9) **5**(80), 1113–1145 (1921)], gives all embedding properties; new properties, relating to the neighborhoods of the third and fourth order of a point of the embedded manifold, are given.
Author's summary.

de Rham, Georges. Sur la reductibilité d'un espace de Riemann. Comment. Math. Helv. 26, 328–344 (1952).

Parallel displacement along a closed piecewise differentiable curve, beginning and ending at a point λ, in a Riemannian space V produces a linear transformation of the vector of the tangent space $T(\lambda)$ at λ, and the set of all such transformations constitute the homogeneous holonomic group Ψ of V. The space V is said to be reducible if Ψ is reducible. The main object of this paper is to show that if V is simply connected and complete, and if $T(\lambda)=\sum_{i=1}^{k}T_i$, where each T_i is invariant under Ψ, then there exist k Riemannian spaces $V_1, \cdots, V_k$, with homogeneous holonomic groups $\Psi_1, \cdots, \Psi_k$ such that the direct product of the V_i is isometric to V, and Ψ is isomorphic to the direct product of the Ψ_i. It is further shown that when V is simply connected and complete, it is isometric to an essentially unique product $V_1\times\cdots\times V_k$, where at most one V_i is Euclidean, and the others are irreducible. An appendix gives a new proof of the equivalence of the various characterizations of a complete space [cf. H. Hopf and W. Rinow, Comment. Math. Helv. 3, 209–225 (1931)].

W. V. D. Hodge (Cambridge, England).

Norden, A. P. On an interpretation of a space with degenerate metric. Doklady Akad. Nauk SSSR (N.S.) 50, 57–60 (1945). (Russian)

The paper is really concerned with an n-dimensional affine space with a Weyl metric which is singular of rank $n-1$. In this case since $g_{\alpha\beta}g^{\beta\gamma}=0$, $g^{\beta\gamma}$ must be of rank 1 and therefore $g^{\beta\gamma}=g^{\beta}g^{\gamma}$, where g^{β} is a contravariant vector. From the conditions for a Weyl metric it follows that $\nabla_\alpha g_{\beta\gamma}=2q_\alpha g_{\beta\gamma}$ and $\nabla_\alpha g^{\beta}=-q_\alpha g^{\beta}$. Introducing a canonical coordinate system in which $g^{\beta}=\delta^{\beta}{}_{(n)}$, the g's and Γ are found to satisfy the following conditions: (1) g_{ij} are functions of $u^1, u^2, \cdots, u^{n-1}$ ($i, j=1, \cdots, n-1$; $\alpha, \beta=1, \cdots, n$); (2) $q_\alpha=\delta_\alpha{}^{(n)}$; (3) $\Gamma^i{}_{jk}=\{{}^{i}_{jk}\}$; (4) $\Gamma^k{}_{\alpha n}=0$, $\Gamma^n{}_{n\alpha}=-q_\alpha$. A space with this structure is denoted by D_n. The coordinates $u^1, u^2, \cdots, u^n$ are taken to define a hyperplane L_{n-1}. A point of this hyperplane is defined by a covariant vector v_α determined up to a scalar multiplier. A k-dimensional plane L_k is determined by a tensor $a_{\alpha_1\alpha_2\cdots\alpha_{k+1}}$ or equivalently by

$$a^{\alpha_{k+2}\cdots\alpha_n}=\epsilon^{\alpha_1\cdots\alpha_n}a_{\alpha_1\cdots\alpha_{k+1}}.$$

A point is then incident with L_k if $a^{\alpha_{k+2}\cdots\alpha_n}v_{\alpha_n}=0$. The space L_{n-2} determined by g^α is called improper and all its incident points are improper points. All proper points may be normalized so that $g^\alpha v_\alpha=1$. Then the distance between any two proper points is given by $l^2=\sum_{i=1}^{n-1}(v_i-w_i)^2$ so that every L_{n-1} is a Euclidean space of $n-1$ dimensions. A similar investigation is made for a space D_n of constant Riemannian curvature.

M. S. Knebelman (Pullman, Wash.).

Mutō, Yosio. On the affinely connected space admitting a group of affine motions. Proc. Japan Acad. 26, nos. 2–5, 107–110 (1950).

A highly dubious paper dealing with affine collineations. The author attempts to prove that there exists no affinely connected space V_n with a group of collineations G_r where $n^2<r<n^2+n$. He assumes that there is a group for which $r=n^2$ which implies that there are precisely n linearly independent equations among the conditions of integrability of the equations of infinitesimal collineations. From these the author concludes by "some calculations not so simple" that either $R^\lambda{}_{\alpha\beta\gamma}=A_\alpha(\delta_\beta{}^\lambda A_\gamma-\delta_\gamma{}^\lambda A_\beta)$, or that the space is flat. Assuming that these calculations are correct, all that is proved is that an affine space whose curvature tensor has the above form admits G_{n^2} as a group of collineations.

M. S. Knebelman (Pullman, Wash.).

Reeb, Georges. Quelques propriétés globales des géodésiques d'un espace de Finsler et des variétés minima d'un espace de Cartan. Colloque de Topologie de Strasbourg, 1951, no. II, 9 pp. La Bibliothèque Nationale et Universitaire de Strasbourg, 1952.

The notion of a Finsler space is generalized to that of Cartan structure (F_p): a manifold V_n, on which an "integrand" $f(x, u_p)$ is given where u_p runs through the p-vectors at x (for Finsler spaces $p=1$), with the usual properties. Let $V_n{}^p$ be the space of p-forms on V_n; it is a fiber space over V_n, projection P. A basic role is played by the form Θ_p on $V_n{}^p$, whose value at a point (x, w) of $V_n{}^p$ ($x\in V_n$, w a p-form at x) is $P^*(w)$, and by the form $\Omega_p=d\Theta_p$. The function f (cf. above) gives rise to a function f^* on $V_n{}^p$; by means of $f^*=1$ one gets a $(2n-1)$-submanifold W_{2n-1} of $V_n{}^p$, which is an $(n-1)$-sphere bundle over V_n (for $p=1$ or $n-1$). The forms Θ_p and Ω_p determine on W_{2n-1} the forms $\Theta_p(f)$ and $\Omega_p(f)$; the latter, for $p=1$, is Cartan's integral invariant. The differential equation for the calculus of variations problem defined by f, are the characteristic equations of $\Omega_p(f)$; the solution manifolds of the latter, in W_{2n-1}, project into the extremals on V_n. Applications (non-existence of compact transversal manifolds in W_{2n-1}, generalization of mean curvature).

H. Samelson (Princeton, N. J.).

Reeb, Georges. Sur certaines propriétés globales des trajectoires de la dynamique, dues à l'existence de l'invariant intégral de M. Elie Cartan. Colloque de Topologie de Strasbourg, 1951, no. III, 7 pp. La Bibliothèque Nationale et Universitaire de Strasbourg, 1952.

On a manifold W_{2n-1} let E be a line-element-field, and let the differential-1-form π, of maximal class (i.e., $(d\pi)^{n-1}\neq 0$), be a (relative) integral invariant of E; this occurs, e.g., in dynamical systems. Theorem I: There exists no compact $(2n-2)$-manifold W_{2n-2} transversal to E. Proof: $0=\int_{W_{2n-2}}(d\pi)^{n-1}\neq 0$. This explains the necessity of considering surfaces of section with boundary. It is then further assumed that W_{2n-1} is compact and that the trajectories of E are all closed, giving a fibration of W_{2n-1}, with base space W_{2n-2}; natural examples are cited. $d\pi$ induces on the base W_{2n-2} a 2-form Ω, which is closed and has $\Omega^{n-1}\neq 0$; this means that W_{2n-2} is symplectic; Ω is the characteristic class of the fibration. Consequences, for the Betti numbers, are stated. A compact manifold V_n is called of type F if it admits a Finsler structure such that the manifold W_{2n-1} of unit vectors is of the type described above with respect to geodesic flow (i.e., mainly, all geodesics are closed). One can then compute the Betti numbers of the space W_{2n-2} of geodesics in terms of those of V_n. Theorem: The odd-dimensional Betti numbers of an F-manifold are even; the first is 0; if n is even, then the Euler characteristic is non-zero.

H. Samelson (Princeton, N. J.).

Reeb, Georges. Variétés symplectiques, variétés presque-complexes et systèmes dynamiques. C. R. Acad. Sci. Paris 235, 776–778 (1952).

A dynamical system (D. S.) is a manifold V_{2n+1}, on which an exterior 2-form A of rank $2n$ is given; a D. S. for which $dA=0$, is called D. S. I. (I refers to "integral invariant"); the definitions are suggested by the phase-time spaces of mechanical systems. A defines a vector field E_1, such that the interior product of E_1 and A is 0. If V_{2n+1} is replaced

by V_{2n}, one gets almost complex (A. C. M.) and symplectic (S. M.) manifolds. Several propositions connecting these concepts are given. E.g.: (1) If f is a map of rank $2n$ of W_{2n} into the D. S. V_{2n+1}, with $f(W_{2n})$ transversal to E_1, the W_{2n} is an A. C. M. with defining form $f^*\Lambda$. (2) A compact V_{2n} with Euler characteristic 0, which can be imbedded in Euclidean R_{2n+1}, is an A. C. M. (construct a vector field in R_{2n+1}, transversal to V_{2n}; it induces a D. S. structure in R_{2n+1}; now apply (1)). *H. Samelson* (Princeton, N. J.).

Libermann, Paulette. Formes différentielles extérieures sur une variété V_{2n}. Colloque de Topologie de Strasbourg, 1951, no. VIII, 7 pp. La Bibliothèque Nationale et Universitaire de Strasbourg, 1952.

In this lecture the results of an earlier note [Ehresmann and Libermann, C. R. Acad. Sci. Paris **229**, 697–698 (1949); these Rev. **11**, 251] are completed. The first part treats Euclidean R^{2n}. In addition to the customary adjoint operator with respect to a positive quadratic form F one defines an adjoint operator with respect to a skew 2-form Ω of rank $2n$. The main result here is that, if F and Ω are exchangeable in Ehresmann's sense, then the two adjoints of the operator L ($\varphi\rightarrow\varphi\wedge\Omega$) are identical (and so do not depend on F). Then follows the definition of effective and of simple forms, class of a form, the theorem on decomposition in simple forms [cf. Eckmann and Guggenheimer, ibid. **229**, 489–491 (1949); these Rev. **11**, 212], Lepage's theorem. The second part treats a manifold V_{2n}, with a (closed) 2-differential form Ω of rank $2n$, and Riemannian metric F, exchangeable with Ω. The adjoints of L are of course still identical, but there are two codifferentials δ and $\dot{\delta}$. Theorems describing the behavior of class under differentiation are given. If a form φ is harmonic with respect to Ω (i.e., $d\varphi=\dot{\delta}\varphi=0$), then its simple components are also harmonic with respect to Ω, and so is $\varphi\wedge\Omega$. Such a form can be ~ 0 without vanishing identically. *H. Samelson.*

Guggenheimer, Heinrich. Ueber kählersche und symplektische Differentialalgebren. Tôhoku Math. J. (2) **4**, 157–171 (1952).

A study of the algebraic properties of forms on manifolds on which there exists a 2-form Ω of maximum rank everywhere (almost-complex manifolds). The definition of the operator $*$ requires that a metric be given, and since the properties assumed in the case $d\Omega=0$ (the symplectic case) are equivalent to assuming that the manifold is Kählerian, the topological results obtained are well-known.

W. V. D. Hodge (Cambridge, England).

Chaki, M. C. On a non-symmetric harmonic space. Bull. Calcutta Math. Soc. **44**, 37–40 (1952).

An example of a Riemannian space which is harmonic without being symmetric in the Cartan sense is given. More general forms having this property and including the present example as a special case have been given by E. M. Patterson [J. London Math. Soc. **26**, 238–240 (1951); **27**, 102–107 (1952); these Rev. **12**, 858; **13**, 986]. *A. G. Walker.*

Kikuchi, Shigetaka. On the theory of subspace in a Finsler space. Tensor (N.S.) **2**, 67–79 (1952).

The author obtains some generalized Frenet formulae. He also defines the notions of totally extremal and totally geodesic hypersurfaces, and obtains conditions upon the surrounding Finsler space in order that there may exist such hypersurfaces immersed in it. *E. T. Davies.*

Hiramatu, Hitosi. On projective collineations in a space of hyperplanes. Tensor (N.S.) **2**, 1–14 (1952).

A system of hyperplanes is defined by Pfaffian equations of the form

$$(1)\qquad u_i dx^i=0,\quad du_j-\Gamma_{jk}(x,u)dx^k=0.$$

If another set of functions $\bar{\Gamma}_{ij}(x,u)$ is related to the given ones by

$$\bar{\Gamma}_{jk}(x,u)=\Gamma_{jk}(x,u)+\lambda_j(x,u)u_k+\lambda_k(x,u)u_j,$$

the set of hyperplanes corresponding to (1) with Γ_{jk} replaced by $\bar{\Gamma}_{jk}$ is said to be projectively related. A point transformation is a projective collineation if it transforms one hyperplane into another which is projectively related to the original. In this paper the author studies infinitesimal point transformations of the form

$$(2)\qquad \bar{x}^i=x^i+\xi^i(x)dt$$

and expresses the necessary and sufficient condition upon the ξ^i and the λ_j in order that (2) may determine an infinitesimal projective collineation. The main tool in the investigation is the Lie derivative. *E. T. Davies.*

Kawaguchi, Akitsugu. On the theory of non-linear connections. I. Introduction to the theory of general non-linear connections. Tensor (N.S.) **2**, 123–142 (1952).

Non-linear connections have been studied before, e.g., by Friesecke [Math. Ann. **94**, 101–118 (1925)], Bortolotti [Ann. of Math. (2) **32**, 361–377 (1931)], Mikami [Jap. J. Math. **17**, 541–568 (1941); these Rev. **3**, 311], Cartan, Schouten [Rend. Circ. Mat. Palermo **50**, 142–169 (1926)], and the author [Proc. Imp. Acad. Tokyo **8**, 340–343 (1932); **9**, 351–354 (1933)], but a general theory had not been developed. The present paper (and others to come) is intended to fill this gap, and to supply tools for a new attack on Finsler and Cartan spaces, areal spaces, and on spaces of general paths, spreads, etc.

The covariant differential δv^I of a contravariant vector v^I in any set of associated linear spaces (fibre bundle) $E_m(x)$ is defined by

$$\delta v^I=dv^I+\omega_i^I(x,v^*)dx^i,$$

where the $\omega_i^I(x,v^*)$ are functions of x and of $v^1,\cdots,v^m$, homogeneous of degree one in $v^1,\cdots,v^m$. The postulate $\delta(uv)=u\cdot\delta v+\delta u\cdot v$ for general (i.e., non-contracted) products defines the covariant differential of any contravariant tensor or density. In particular,

$$\delta T^{IJ}=dT^{IJ}+\omega_i^{IJ}(x,T^{**})dx^i,\quad \delta\mathfrak{p}=d\mathfrak{p}-\mathfrak{p}\omega_i(x)dx^i,$$

where

$$\omega_i^{IJ}(x,T^{**})=\omega_i^I(x,T^{*J})+\omega_i^J(x,T^{I*}),\quad \omega_i(x)=\omega_i^J(x,\delta_J^*).$$

Similarly, δw_J is defined by

$$\delta w_J=dw_J-\omega'_{Ji}(x,w_*)dx^i.$$

Since W_J can be considered as a contravariant $(m-1)$-vector density, ω' can be expressed in terms of ω.

Although $A_J^I=0$ (A_J^I is the unit tensor), the formula $A_I^J\delta(v^Iw_J)=\delta(v^Iw_I)$ cannot hold unless the connection is linear. The same applies to $\delta(u^I+v^I)=\delta u^I+\delta v^I$.

Three quantities enter that do not occur in the classical theories: the alinearity scalars

$$\chi_{Ji}(x,v_*)=\omega^L{}_{Li}(x,\delta_L^*)-\omega^L{}_{Li}(x,2\delta^{*K}_{[LJ]}v_K)\quad(\omega^I{}_{Ji}=\partial\omega_i^I/\partial v^J),$$

the alinearity tensor

$$F^I{}_{KJi}=\partial\omega^I{}_{Ki}/\partial v^J,$$

and the contraction tensor

$$T^I{}_{Ji}(x, v^*, w_*) = \omega^I{}_{Ji}(x, v^*) - \omega'^I_{Ji}(x, w_*) \quad (\omega'^I_{Ji} = \partial\omega'{}_{Ji}/\partial w_I).$$

$T^I{}_{Ji}$ vanishes if and only if the connection is linear.

In order to avoid some of the complications that arise from the non-linearity of the connection, "relative covariant differentials" are also introduced:

$$\delta_{(v)} w^I = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \{\delta(v^I + \epsilon w^I) - \delta v^I\} = dw^I + \omega^I{}_{Ji}(x, v^*) dx^i.$$

These differentials are discussed in some detail.

The paper ends with an application to spaces of curve elements of higher order. *A. Nijenhuis.*

Kawaguchi, Akitsugu, and Katsurada, Yoshie. On a connection in an areal space. Jap. J. Math. **21** (1951), 249–262 (1952).

The authors consider the differential geometry of a space in which the fundamental invariant is an integral $\int_{(m)} f(x^i, \partial x^i/\partial u^\alpha) du^1 \cdots du^m$ extended over a region of the subspace of dimension m which is immersed in an n-dimensional space X_n, and is given parametrically by the set of equations $x^i = x^i(u^\alpha)$, $i = 1, 2, \cdots, n$, $\alpha = 1, 2, \cdots, m$. For formal simplicity they confine themselves to the case $m = 2$. Vectors, tensors, and parameters of connection are all functions of the x^i and of $\partial x^i/\partial u^\alpha$. After introducing a four-index metric tensor, the authors proceed to a generalization of the notion of euclidean connexions as given by E. Cartan for Finsler spaces [Les espaces de Finsler, Hermann, Paris, 1934]. They introduce certain restrictions which correspond to the fundamental postulates given by Cartan. With the help of these restrictions they have been able to express the two sets of connection parameters occurring in the expression of an absolute differential $\delta v^i = dv^i + \Gamma^i{}_{jk} v^j dx^k + C^{i\alpha}_{j\cdot k} v^j dp_\alpha{}^k$ in terms of generalized symbols of Christoffel which are in turn deduced from the fundamental integrand functions.

E. T. Davies (Southampton).

Kawaguchi, Akitsugu, and Tandai, Kwoichi. On areal spaces. V. Normalized metric tensor and connection parameters in a space of the submetric class. Tensor (N.S.) **2**, 47–58 (1952).

The subject matter of this paper depends on previous communication by one of the authors on the same subject [Kawaguchi, Tensor (N.S.) 1, 14–45 (1950); 1, 67–88, 89–103 (1951); these Rev. 12, 536; 13, 384, 385]. Some further results are obtained on spaces of the submetric class. It is shown that in any regular space of the submetric class there can always be defined a symmetric tensor of the second order called the normalized metric tensor. If the submetric space happens to be of the restricted type known as the metric class, then this normalized metric tensor coincides with the ordinary metric tensor of the space. The introduction of this normalized metric tensor leads to a considerable simplification of the theory of connections in spaces of the submetric class. The authors also introduce what they call an ecmetric tensor whose identical vanishing characterizes spaces of the metric class. *E. T. Davies* (Southampton).

Ide, Saburo. On the theory of curves in an n-dimensional space with the metrics $s = \int (A_i x''^i + B)^{1/p} dt$. II. Tensor (N.S.) **2**, 89–98 (1952).

[For part I see Tensor **9**, 25–29 (1949); these Rev. **12**, 206.] By the use of a suitable process of covariant differentiation, the author develops the theory of curves in spaces with a metric given by $s = \int (A_i x''^i + B)^{1/p} dt$. The resulting theory has formal similarity to the classical Frenet equations. The differential is obtained as follows. Provided that $p \neq 3/2$, put $G_{ij} = 2(A_{i(j)} - A_{j(i)})$. Let $\Gamma^i = \frac{1}{2}(2A_{jk} x'^k - B_{(j)}) G^{ji}$. By definition $\delta v_i = dv_i - \Gamma^j{}_{(i)(k)} v_j dx^k$. Provided that $p \neq 3$, put $g_{ij} = (p-3)^{-1}(A_{i(j)} + A_{j(i)})$. Finally define

$$\Lambda_j{}^i = \Gamma^i{}_{(j)(k)} dx^k + \tfrac{1}{2} g^{ih} \delta g_{jh}.$$

Then the desired covariant differential is $Dv^i = dv^i + \Lambda_j{}^i v^j$.

C. B. Allendoerfer (Seattle, Wash.).

***Anđelić, Tatomir, P. Tenzorski račun.** [Tensor calculus.] Naučna Knjiga, Belgrade, 1952. viii+319 pp.

This is a first course in tensor calculus and its applications to differential geometry, mechanics, elasticity, and fluid flow. No affine geometry is considered. The notations and the placement of indices are careful; the Schouten style is used except for the Christoffel symbols where the brackets still persist. Some computations are perhaps longer than is necessary (p. 164) because the author works with the coefficients rather than with the corresponding scalar forms. There are occasional excursions into direct notation. On the whole, the book exemplifies the type of tensor text that has stabilized itself in the course of the past ten years or so.

A. W. Wundheiler (Chicago, Ill.).

Gurevič, G. B. Complete systems of symmetric and skew-symmetric tensors. Mat. Sbornik N.S. **27**(69), 103–116 (1950). (Russian)

Certain results of Lie algebra [Gurevič, C. R. (Doklady) Acad. Sci. URSS (N.S.) **45**, 47–49 (1944); Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 403–416 (1949); these Rev. **7**, 110; **11**, 156] are used to devise a simple combinatorial rule to generate the different possible types of trivectors (skew-symmetric trivalent tensors, or cubic alternating forms) in a linear vector space of dimensions $r \leq 7$, and with field of characteristic 0 or >3.

Let us call two k-numbers $a = (a_1, \cdots, a_k)$, $b = (b_1, \cdots, b_k)$, comparable if $a_i \leq b_i$ or $a_i \geq b_i$. If by reordering the components of a or b they can be made comparable, the author defines them to agree. Now represent the alternating product of base vectors, say $[e_1 e_2 e_3]$, by the corresponding numerals [123]. Then if we form the sum of all possible products such that no two products "agree", and eliminate forms which are merely renaming of others, we automatically obtain the type leading trivectors, or canonical forms.

The results obtained reproduce those originally obtained by Schouten [Rend. Circ. Mat. Palermo **55**, 137–156 (1931)] by an involved geometric method, for $r \leq 7$, and agree with those of the reviewer [Dissertation, Mass. Inst. Tech., 1940] for $r \leq 8$, obtained by a less simple but more general combinatorial method. The author also neatly obtains the well-known results for bivectors and symmetric bivalent tensors.

L. C. Hutchinson (Boston, Mass.).

NUMERICAL AND GRAPHICAL METHODS

★**Desyatiznačnye tablicy logarifmov kompleksnyh čisel i perehoda ot dekartovyh koordinat k polyarnym. Tablicy funkcil** $\ln x$, $\operatorname{arctg} x$, $\frac{1}{2}\ln(1+x^2)$, $\sqrt{1+x^2}$. **[Ten-place tables of logarithms of complex numbers and of the transformation from cartesian to polar coordinates. Tables of the functions** $\ln x$, $\operatorname{arctg} x$, $\frac{1}{2}\ln(1+x^2)$, $\sqrt{1+x^2}$.**]** Izdat. Akad. Nauk SSSR, Moscow, 1952. 116 pp. (1 plate). 10.80 rubles.

These tables contain $\ln x$, for $x=0(0.001)10$, and $\frac{1}{2}\ln(1+x^2)$, $\arctan x$, and $(1+x^2)^{1/2}$, for $x=0(0.001)1$, all with ten decimal places. Both first and second differences are listed. An insert contains $\frac{1}{4}x(1-x)$ for $x=0(0.001)1$, 4D for use in quadratic interpolation, as well as $\ln 10^n$, $n=1(1)25$, 12D.

Brower, W. B., and Lassen, R. H. Additional values of $C(k)$. J. Aeronaut. Sci. **20**, 148–150 (1953).

Tabulation, to seven decimals, of Theodorsen's function $C(k)=F(k)+iG(k)$ where

$$F=\frac{J_1(J_1+Y_0)+Y_1(Y_1-J_0)}{(J_1+Y_0)^2+(Y_1-J_0)^2},\quad G=-\frac{Y_1Y_0+J_1J_0}{(J_1+Y_0)^2+(Y_1-J_0)^2}$$

for $k=1.00, 1.02, 1.04, \cdots, 5.00, 5.10, \cdots, 10.00, 10.50, \cdots, 16.00$. *E. Reissner* (Cambridge, Mass.).

Oettinger, Anthony G. Programming a digital computer to learn. Philos. Mag. (7) **43**, 1243–1263 (1952).

The author is concerned with the problem of programming a digital machine to "learn" by experience. He discusses in detail two examples of such programs. The first of these examples is what he terms a shopping program and in effect consists of describing in a matrix whether a shop i has an article j. At the beginning the machine is programmed to choose shops at random and to search each shop until it finds the desired article. It then stores this knowledge of where to find the article. In the future it will then first consult this "prior experience" to see if it knows where the article is to be found before it starts. If the machine finds it has previously obtained this article in a given shop, it can then go directly there without the need for a random search. The second example is a so-called response-learning program. In this program the author "conditions" responses in the machine and causes it to form "habits". Both examples were worked out in detail and the programs carried out on the EDSAC, the digital computer in Cambridge University. *H. H. Goldstine* (Princeton, N. J.).

Tallqvist, Hj. Ein neues Multiplikations- und Divisions-Verfahren. Soc. Sci. Fenn. Comment. Phys.-Math. **16**, no. 4, 3 pp. (1952).

Mysovskih, I. P. On the convergence of Newton's method for a real equation with conditions of Cauchy type. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 756–759 (1952). (Russian)

Let $P(x)$ be a real twice differentiable function satisfying the following properties: $P(x_0)P'(x_0)>0$ $\{P(x_0)P'(x_0)<0\}$, $P''(x)$ exists in the interval $I[x_0-\Lambda, x_0]$ $\{[x_0, x_0+\Lambda]\}$, $|P''(x)|\leq K$ and $|1/P'(x)|\leq B$ on I, $|P(x_0)|<\eta$, and $h=B^2K\eta\leq 4$. Then the equation $P(x)=0$ has a unique solution to which the sequence defined by

$$x_{n+1}=x_n-P(x_n)/P'(x_n),\qquad n=0, 1, \cdots,$$

converges. [Cf. Ostrowski, Mat. Sbornik N.S. **3**(45), 253–258 (1938).] An example is given to show that the constant 4 in $h\leq 4$ cannot be improved. Another similar theorem is also proved. *J. V. Wehausen* (Providence, R. I.).

Cicala, P. Determination of modes and frequencies above the fundamental by matrix iteration. J. Aeronaut. Sci. **19**, 719–720 (1952).

This article deals with the well-known fact that, given a matrix D, a matrix S can be found so that the second eigenvalue of D is the greatest eigenvalue of DS. The matrix S depends on the first eigenrow $-p_1'$ of D. In practice, the first eigenrow is only approximatively known. This gives an error in the second eigenvalue of D, calculated from DS. The best possible choice for S is $S=I-\lambda_1\bar{q}_1\bar{p}_1'$, where I is the unit matrix, λ_1 is the first eigenvalue, $\bar{q}_1$ and $\bar{p}_1'$ are approximations to the first eigencolumn and eigenrow. *W. H. Muller* (Amsterdam).

Warga, J. On a class of iterative procedures for solving normal systems of ordinary differential equations. J. Math. Physics **31**, 223–243 (1953).

Let z and $f(z, t)$ be n-dimensional vectors, and consider the system (1) $\dot{z}=f(z, t)$ of ordinary differential equations. A common idea underlying a great number of known iteration methods for solving initial-value problems for such systems is formulated by proving a theorem which states, in essence, that the solutions of the sequence of initial value problems $\dot{y}_j=G_j(y_j, t)$, $y_j(t_0)=z_0$, tend uniformly, together with their derivatives, to the corresponding solutions of (1) if and only if $G_{j+1}(y_j(t), t)-f(y_j(t), t)$ tends uniformly to zero. Among the procedures that can be formulated in this manner by a suitable choice of the $G_j(y, t)$ are the Taylor expansion procedure, Picard's method, the method of Kantorovitch [Doklady Akad. Nauk SSSR (N.S.) **59**, 1237–1240 (1948); these Rev. **9**, 537], expansion in terms of a parameter, and three methods by Germay [Bull. Soc. Roy. Sci. Liège **15**, 510–513 (1946); **16**, 119–125 (1947); **18**, 3–8 (1949); these Rev. **9**, 435; **10**, 712].

The author's methods lead also to appraisals of the rates of convergence of these procedures. Kantorovitch's technique proves to be the most rapidly convergent one, but it is analytically very complicated. A somewhat less rapidly convergent variant of this method is therefore proposed, in which the successive iterants can be found by quadratures. Finally, a modification of Picard's method is suggested, by means of which the introduction of errors through the use of approximate numerical quadrature formulas can be avoided. *W. Wasow* (Los Angeles, Calif.).

Muhin, I. S. On the accumulation of errors in numerical integration of differential equations. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 753–755 (1952). (Russian)

The author considers the growth of the error in the step-by-step numerical solution of a differential equation of the second order, comparing three methods: 1) Milne's three-ordinate method for a second order equation lacking a first derivative; 2) a method of Muhin, using first, second, and third derivatives; and 3) another method, apparently also due to Muhin, using first and second derivatives. The comparison indicates that Muhin's methods are highly accurate, Milne's quite inaccurate. [Reviewer's note. The calculation was carried to four places in an example for which the

methods used are accurate to about eight places. The comparison therefore pertains only to round-off error and sheds no real light on the relative merits of the three methods.]
W. E. Milne (Corvallis, Ore.).

Zadiraka, K. V. On the construction of two-sided approximations for the characteristic values of a Sturm-Liouville boundary problem. Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 735–738 (1952). (Russian)

By choosing a suitable approximation $m^2(\lambda)$, to $Q(x, \lambda)$, writing $y''+Q(x, \lambda)y=0$ in the form $y''+m^2y=(m^2-Q)y$, and converting this last equation into an integral equation $y=y(0) \cos mx+y'(0) \sin mx/m+T(Q, y)$, the author shows how to obtain approximations for the characteristic values of the associated Sturm-Liouville problems. *R. Bellman.*

***Massau, J. Mémoire sur l'intégration graphique des équations aux dérivées partielles.** G. Delporte, Mons, 1952. 9+391+88 pp.

Reprinted by photo-offset from Annales de l'Association des Ingénieurs sortis des Ecoles spéciales de Gand (2) 23, 95–214 (1900); (3) 1, 135–226, 393–434 (1902); 2, 383–436 (1903); 3, 65–147 (1904). In addition to the work of the title, this volume contains a reprint of another closely related paper: Note sur l'équation des cordes vibrantes, ibid. (3) 4, 65–152 (1905).

Aĭzenštat, N. D. On an estimate of the error in approximate solution of a finite-difference Poisson equation. Mat. Sbornik N.S. 31(73), 485–490 (1952). (Russian)

The author considers the quation

$$\Delta_h u(x, y) = u(x+h, y)+u(x-h, y)+u(x, y+h)+u(x, y-h) -4u(x, y)=h^2f(x, y)$$

with assigned values for $u(x, y)$ on the boundary of a region covered by a square net with mesh length h. His principal result is this: If $v(x, y)$ satisfies the boundary conditions and is such that $|h^{-2}\Delta_h v-f|<g$, for all interior nodal points, then

$$|v-u| \leq gh^2(2^{N+1}-N-2), \quad |v-u| \leq gh^2N^2/2,$$

in which N is the number of steps from the interior point (or points) farthest from the boundary. *W. E. Milne.*

Collatz, L. Fehlerabschätzung bei der ersten Randwertaufgabe bei elliptischen Differentialgleichungen. Z. Angew. Math. Mech. 32, 202–211 (1952). (German. English, French and Russian summaries)

The author extends the boundary maximum theorem for subharmonic functions to apply, under certain restricted conditions, to more general second-order linear differential equations of elliptic type. This result is used to obtain an estimate of the error involved in solving these types of equations approximately by the Trefftz method (using functions which satisfy the differential equation), the Ritz method (using functions which satisfy the boundary conditions only), and by a method due to von Mises (using functions which approximate both the differential equation and boundary conditions simultaneously). Illustrative examples are given. *H. Polachek* (Carderock, Md.).

Grünsch, H. J. Eine Fehlerabschätzung bei der dritten Randwertaufgabe der Potentialtheorie. Z. Angew. Math. Mech. 32, 279–281 (1952).

In this note the author develops an upper bound for the error involved in estimating the solution of the Laplace equation in three dimensions by means of a function which does not necessarily satisfy the given boundary conditions. The result is extended to other elliptic equations.
H. Polachek (Carderock, Md.).

de G. Allen, D. N., and Severn, R. T. The application of relaxation methods to the solution of non-elliptic partial differential equations. II. The solidification of liquids. Quart. J. Mech. Appl. Math. 5, 447–454 (1952).

This is the continuation of an earlier paper [same Quart. 4, 209–222 (1951); these Rev. 13, 287] by the same authors. It uses the methods described in the earlier paper to study the cooling of a substance which passes from liquid to solid state during the process of cooling. The space-time frontier separating the liquid from the solid states, as well as the space-time distribution of temperature, is determined by the use of relaxation. *W. E. Milne* (Corvallis, Ore.).

Bartlett, James H. Iterative procedures and the helium wave equation. Physical Rev. (2) 88, 525–526 (1952).

The determination of the ground state of the helium atom involves the solution of the equation

$$\nabla^2 W+\left(\frac{E-V}{4r}+\frac{1}{4s^2}\right)W=0,$$

in three independent variables x, y, and z. A numerical method is used, based on the replacement of the differential equation by finite-difference equations, and the eigenvalue E computed by applying Liebmann's method to the homogeneous algebraic equations so obtained. With the simple formula, involving at each point only six neighbouring points, the result is not very accurate on a mesh of 180 points. An alternative formula is proposed which has better accuracy but involves 26 neighbouring points. This formula is the three-dimensional extension, with unequal intervals, of the following well-known approximations in one or two dimensions and with equal intervals h.

$$f_{r+1}-2f_r+f_{r-1}=\frac{1}{12}h^2(f''_{r+1}+10f_r''+f''_{r-1}).$$

$$8\sum_1^4{}_s f+2\sum_5^9 f_s-40f_0=\frac{1}{12}h^2\left(100\nabla^2 f_0+10\nabla^2\sum_1^4 f_s+\nabla^2\sum_5^9 f_s\right)$$

where $s=1$ to 4 corresponds to the points $(\pm h, 0)$, $(0, \pm h)$, and $s=5$ to 9 to the points $(\pm h, \pm h)$. The terms on the right are replaced by functional values by using the differential equation. The computations were performed on SEAC at the Bureau of Standards. *L. Fox* (Teddington).

van den Dungen, Frans H. Sur l'intégration numérique des équations différentielles hyperboliques linéaires. C. R. Acad. Sci. Paris 236, 42–43 (1953).

Instead of solving the equation (*) $u_{xx}=c^{-2}u_{tt}$ immediately by finite differences, the author proposes to apply an appropriate transform theorem which will reduce (*) to a set of ordinary differential equations and then solve these by some numerical method, e.g., finite differences.
B. Friedman (New York, N. Y.).

Thornhill, C. K. The numerical method of characteristics for hyperbolic problems in three independent variables. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. 2615 (11,767), 13 pp. (1952).

A general quasi-linear second order partial differential equation for one unknown function of three independent variables is considered. The concepts of and the equations

for characteristic surfaces, bi-characteristic curves, characteristic cones and conoids are discussed for the case of a "hyperbolic" equation. The characteristic relations are derived in detail for the equation of steady supersonic compressible flow in three-dimensional space, and for the equation of unsteady compressible flow in two dimensions. The title of the paper may be misleading because only a brief description is given of how to use the characteristic relations to solve such problems numerically on a hexahedral grid.
E. Isaacson (New York, N. Y.).

Henrici, Peter. Weitere Bemerkung zu $\int e^{b(x+a\cos x)}dx$. Z. Angew. Math. Physik 3, 466–468 (1952).

The integral between the limits $-\infty$ and X is expanded in a series whose terms are products of Bessel and trigonometric functions, valid for $R(b)>0$, real X, and real a in $(-1, 1)$. *P. W. Ketchum* (Urbana, Ill.).

Sharpe, Joseph A., and Fullerton, Paul W. An application of punched card methods in geophysical interpretation. Geophysics 17, 707–720 (1952).

There is a large class of problems in geophysical interpretation which is solved by the integral over a plane surface of the product of an observed field value or an assumed surface density of some characteristic property F by a weighting function W,

$$S(x, y, z)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(u, v, 0)W(x-u, y-v, z)dudv.$$

Use of a trigonal coordinate system and substitution of finite sums for the infinite integrals leads to a particularly useful form, of adequate practical accuracy, for calculation by punched card methods. The geophysical interpretation problems to which this attack applies are listed, and an illustration is given of the calculation of the second vertical derivative of magnetic total intensity by employment of the widely available I.B.M. punched card machines.
Authors' summary.

Schubert, Hans. Über ein gemischtes räumliches Randwertproblem der Potentialtheorie. II. Math. Nachr. 7, 335–338 (1952).

The author carries through in detail the numerical calculation of the downwash velocity from the integral equation for it given in part I [Math. Nachr. 5, 93–110 (1951); these Rev. 13, 130]. *D. Gilbarg* (Bloomington, Ind.).

★Seidel, W. Bibliography of numerical methods in conformal mapping. Construction and applications of conformal maps. Proceedings of a symposium, pp. 269–280. National Bureau of Standards, Appl. Math. Ser., No. 18, U. S. Government Printing Office, Washington, D. C., 1952. $2.25.

Biermann, L., und Billing, H. Moderne mathematische Maschinen. Naturwissenschaften 40, 7–13 (1953).

Williams, F. C., Kilburn, T., and Tootill, G. C. Universal high-speed digital computers: a small-scale experimental machine. Proc. Inst. Elec. Engrs. Part II. 98, 13–28 (1951).

Sutor, Josef. Neue einfache Verfahren der Auswertung und Triangulation von Senkrechtaufnahmen flachen Geländes. Allg. Vermessgs.-Nachr. 1952, 295–311 (1952).

The vertical mapping of a plane terrain is studied and approximation formulae are derived for the distortion of angles and lengths. *E. Lukacs* (Washington, D. C.).

ASTRONOMY

★Krat, V. A. Figury ravnovesiya nebesnyh tel. [Figures of equilibrium of celestial bodies.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 329+x pp. 12.20 rubles.

The main purpose of the book is to present a theory of stratification in stars and to discuss the problem of double stars. The personal contributions of the author concern this problem. The first four chapters deal with general data about stars and planets, potential theory, absolute equilibrium of a mass of gas, and rotation of a compressible liquid. In Chapter V the Maclaurin and Jacobi ellipsoids are considered as well as the Roche model and polytropic configurations. The problem of small deformations of rotating gas masses is discussed in Chapter VI. The discussion begins, however, with the presentation of the Clairaut problem in the form given by Liapounoff. This is followed by Chandrasekhar's method for the case of compressible gases, by its generalisation for the general case of a barotropic rotation, by v. Zeipel's theorem and by analysis of the Carrington-Faye law of the Sun rotation. In the double star problem (Chapter VII) use is made of Roche's approximation, of the tidal problem, and the model of a polytropic double star is developed in more details. Chapter VIII deals with the linear series of figures of equilibrium and the last chapter (IX) with some cosmological problems. The list of references is rather short. *W. S. Jardetzky.*

Ghosh, N. L. On the equilibrium of a thin atmosphere round a heavy central core: spheroidal and anchor-ring configurations. Bull. Calcutta Math. Soc. 44, 22–26 (1952).

Sizova, O. A. On the possibility of capture in the restricted problem of three bodies. Doklady Akad. Nauk SSSR (N.S.) 86, 485–488 (1952). (Russian)

O. Yu. Šmidt [same Doklady (N.S.) 58, 213–216 (1947)] gave a numerical example to show the possibility of capture in the general problem of three bodies. The impossibility of such capture in the circular case of the restricted problem of three bodies was proved by von Zeipel [Bull. Astr. 22, 449–490 (1905)], Hopf [Math. Ann. 103, 710–719 (1930)] and Fesenkov [Astr. J. Soviet Union [Astr. Žurnal] 23, 45–48 (1946); these Rev. 8, 59]. Following the ideas of Šmidt's paper the author gives a numerical example to show the possibility of capture in the hyperbolic case of the restricted problem of three bodies. *E. Leimanis.*

Szebehely, Victor G. On the problem of three bodies in a plane. Math. Mag. 26, 59–66 (1952).

In the present paper a new set of generalized coordinates is introduced by means of which the 12th order system of differential equations of the planar three bodies problem is reduced to one of the 4th order without using a contact transformation. The special case for which the masses of

the three bodies are equal and the attracting forces are proportional to the cubes of the distances is investigated in detail and two sets of particular solutions are obtained. One of these is represented by trochoidal curves, and the other set splits into two, each of which can be considered as a generalization of the Lagrangian collinear solution. (From the author's summary.) *E. Leimanis.*

Merman, G. A. On a criterion of realizability of hyperbolic-elliptic motion in the problem of three bodies. Doklady Akad. Nauk SSSR (N.S.) **85**, 727–730 (1952). (Russian)

The author establishes that, if certain inequalities are satisfied at time $t=0$ in the three-body problem, then as $t\to+\infty$ one particle will recede to infinity, while the distance between the other two particles is uniformly bounded by a constant R. The inequalities require that constants a, R, a_0, a_1 exist such that at time $t=0$ the quantities $r-aR$, r', $r-a_0r_{02}$, $r-a_1r_{12}$, rr'^2-2A, $(m_0+m_1)r_{01}^{-1}-\frac{1}{2}v^2-(m_0+m_1)R^{-1}-BC^{-1}r^{-2}$, $a_0(a-\mu)-a$, $a_1(a-\lambda)-a$ are positive; here m_0, m_1, m_2 are the three masses, M is their sum, $\lambda=m_1(m_0+m_1)^{-1}$, $\mu=m_0(m_0+m_1)^{-1}$, r is the distance from m_2 to the center of mass of m_0 and m_1 and v is the speed of m_2 relative to this point as origin, r_{ij} is the distance from m_i to m_j,

$$C=(r'^2-2Ar^{-1})^{\frac{1}{2}},\quad 2B=m_2[2(m_0+m_1)R]^{\frac{1}{2}}E,$$
$$E=\mu a_0^3+\lambda a_1^3+a_0a_1(a_0^2+a_0a_1+a_1^2).$$

It is pointed out that this result can be regarded as an improvement on a similar one established by Hil'mi [same Doklady (N.S.) **78**, 653–656 (1951); these Rev. **13**, 789].

W. Kaplan (Ann Arbor, Mich.).

Agekyan, T. A. On the coplanarity of the orbits of triple stars. Akad. Nauk SSSR. Astr. Žurnal **29**, 219–224 (1952). (Russian)

The author develops two different statistical methods for testing possible correlation between inclinations (to the celestial sphere) of the two orbital planes of triple visual systems. Taking into account the effects of observational selection, the author arrives at a conclusion that the data available at present reveal no significant trace of co-planarity of orbital planes in multiple systems—which should once have been complete (as, for instance, in the solar system) if the components of multiple systems are generically related. The effective absence of any significant correlation at the present time indicates, therefore, that either such systems must be extremely old, or their components had no common origin. *Z. Kopal* (Manchester).

Heinrich, Wladimír Wáclav. On certain functional solutions of the satellite problem of three bodies. Acta Math. **88**, 1–75 (1952).

In order to solve the satellite problem of the three bodies, Sun, Earth and Moon, the author starts with a different formulation of the problem than that used by authors who have dealt with the Lunar theory up to the present day. Start from a planetary problem, assuming that the Moon revolves around the Sun of mass $m=1$ and is disturbed by the Earth. Suppose that in the first approximation the Moon, of zero mass when undisturbed, is moving in an ellipse of eccentricity approximately 1/400, and the Earth in a circular orbit, both revolving around the Sun with the same angular velocity $n=n'$, and thus keeping the same starting mean length.

When a rotating system with angular velocity n' is introduced, the Earth becomes a fixed point in this system, but the original planet Moon changes into a satellite, describing a small closed curve around the Earth as its centre, the period being a year instead of a month. The author then studies the analytical continuation of this curve—just in the same way as the classical theory has studied that of an originally fixed, rotating or distorted planetary ellipse around the Earth. He thus obtains huge classes of short periodic and secular particular integrals of the satellite problem in question.

In this new formulation of the Lunar problem the Earth plays the part of the disturbing body, instead of the Sun in the classical theory. This enables the author to choose the mass of the Earth 1/350,000 as a disturbing parameter μ instead of 1/400 in the classical theory.

However, in this new re-formulation of the Lunar problem the author is faced with two impossibilities within the meaning of the classical theory, namely: (i) how to pass from the heliocentric to the geocentric orbit so as to change the original planet into a satellite; (ii) how to set the satellite in motion around the Earth so as to acquire the velocity of our real Moon. The author shows that the resolution of the first impossibility reduces to a fitting passage from heliocentric to geocentric coordinate system, while that of the second to a passage from an identically vanishing Jacobian-Hessian to a determinant different from zero. This last achievement is essential in proving existence of periodic solutions according to Poincaré. *E. Leimanis.*

Merman, G. A. On the radius of convergence of Hill's series. Akad. Nauk SSSR. Byull. Inst. Teoret. Astr. **5**, 185–198 (1952). (Russian)

The intermediate orbit known as the variational curve was obtained by Hill [Coll. math. works, vol. 1, Carnegie Inst. of Washington, 1905, pp. 284–335] as a particular periodic solution of a certain limiting case in the planar restricted problem of three bodies. In the case of the motion of the Moon, Hill represented this solution in powers of a parameter $m=0.080848933\cdots$, the ratio of the mean synodic month to the mean sidereal year. The absolute and uniform convergence of Hill's series was proved by Lyapunov [Trudy Otd. Fiz. Nauk Obšč. Lyub. Estestvozn. **8**, 1–23 (1896)] for $|m|\leqq 1/7$ and independently by Wintner [Math. Z. **30**, 211–227 (1929)] for $|m|\leqq 1/12$. Proskurin [Akad. Nauk SSSR. Byull. Inst. Teoret. Astr. **4**, 169–205 (1949); these Rev. **11**, 466] showed that Brown's analytical theory was incapable of representing the complicated motion of Jupiter's eighth satellite, and was led [ibid. **4**, 341–354 (1950)] to the introduction of Hill's variational curve as a suitable intermediate orbit. In this case the numerical value of m is $m=-0.1461537\cdots$. In order to justify this case the author improves Lyapunov's result, by an extension of his method, and shows the convergence of Hill's series for $|m|\leqq 0.18$. *E. Leimanis* (Vancouver, B. C.).

Camm, G. L. Self-gravitating star systems. II. Monthly Not. Roy. Astr. Soc. **112**, 155–176 (1952).

[For part I see same Not. **110**, 305–324 (1950); these Rev. **12**, 754.] In this paper the following differential equations for describing the density distribution in a spherically symmetrical stellar system are proposed:

$$\text{(case 1)}\qquad \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right)=\frac{e^{-\psi}}{1+\alpha r^2},$$

$$\text{(case 2)}\qquad \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right)=-\frac{\psi^n}{1+\alpha r^2}\quad(\alpha\text{ and }n\text{ constants}),$$

where $e^{-\psi}$ (in case 1) and ψ^n (in case 2) are proportional to the stellar density at distance r. These are the forms which Poisson's equation takes when a solution of Liouville's equation is assumed which is either an exponential (case 1) or a power (case 2) of the integral of the equations of motion which is quadratic in the velocities. This latter integral, I, is a combination of the energy integral (E=constant) and the transverse angular momentum integral (r^2T=constant, T being the transverse velocity): $I=E+\beta r^2T$. In (1) and (2) α is related to the constant β in this integral. Radial star streaming requires α to be positive. Four cases arise depending on whether we retain α in (1) and (2) or set it equal to zero. The author considers these four cases and believes that case (2) with $n=4$ and α finite represents the observed distribution stellar density in globular clusters.

S. Chandrasekhar (Williams Bay, Wis.).

Osterbrock, Donald E. The time of relaxation for stars in a fluctuating density field. Astrophys. J. 116, 164–175 (1952).

The author develops a statistical method, similar in principle to the one used by Chandrasekhar and Münch [Astrophys. J. 115, 103–123 (1952); these Rev. 13, 786] for calculating the brightness fluctuations in the Milky Way, capable of defining the relaxation time of a star moving through a fluctuating density field. The time required for the star to change its direction of motion significantly because of gravitational interaction with matter is expressed in the form of a closed formula (equation (38) of the paper), which can be applied to practical cases. The fluctuating density field is then regarded as representing a simplified model of the interstellar medium; and the computed time of relaxation for this case shows that low-velocity stars can have their velocities appreciably altered by gravitational interaction with the interstellar medium in the course of some 10^9 years.

Z. Kopal (Manchester).

Burbidge, G. R., and Burbidge, E. Margaret. The equation of transfer and the residual intensities in spectrum lines. Astrophys. J. 116, 185–202 (1952).

The equations of transfer for spectral lines arising from the second quantum level are set up (taking account of certain cyclic processes) and solved (by Chandrasekhar's method) for a frequency inside of the respective line and in the neighboring continuum. Criteria for the appearance of emission at any frequency inside of the line are obtained and discussed, and the results are compared with those obtained previously by Underhill [Astrophys. J. 110, 340–354 (1949); these Rev. 11, 409]. The theory was then applied to calculate numerically the emission at the cores of the Balmer hydrogen lines $H\alpha$, $H\beta$, and $H\gamma$ for an adopted range of stellar temperatures and electron pressures. The results (obtained by ignoring the Stark effect) show that, at high temperatures, departures from thermodynamic equilibrium to be expected in stellar atmospheres are likely to induce emission; and it is found that the conditions for emission are relatively insensitive to a wide range of the coefficients of line absorption, continuous absorption, and electron scattering.

Z. Kopal (Manchester).

RELATIVITY

Rickayzen, G., and Kurşunoğlu, B. Unified field theory and Born-Infeld electrodynamics. Physical Rev. (2) 89, 522–523 (1953).

Pastori, Maria. Sulle equazioni del campo elettromagnetico nell'ultima teoria di Einstein. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 12, 302–307 (1952).

Einstein's last unitary theory is expressed in terms of a non-symmetric fundamental tensor g_{jk} and a non-symmetric connection $\Gamma^i{}_{jk}$, subject to conditions which ensure that the tensor derivatives of g_{jk} and of the skew-symmetric tensor ϵ_{ijk} both vanish identically. It is proved (1) that the neutral divergence of any skew-symmetric tensor equals the complete neutral curl of the conjugate tensor, and (2) that the neutral divergence of the skew-symmetric part of g^{jk} vanishes identically, where the neutral derivative of a tensor means the tensor derivative taken with the symmetric part of $\Gamma^i{}_{jk}$. The first equation of the electromagnetic field follows by taking the electromagnetic tensor F_{jk} to be the conjugate tensor of g^{jk} and combining (1) and (2). The second equation of the field merely expresses that the electric current vector equals the neutral divergence of the conjugate tensor of g_{jk}.

A. J. McConnell (Dublin).

Thiry, Yves. Sur une généralisation du problème de Schwarzschild à une théorie unitaire. C. R. Acad. Sci. Paris 235, 1480–1482 (1952).

The author gives the detailed form of the field equations of a five-dimensional unified field theory previously discussed [J. Math. Pures Appl. (9) 30, 275–316, 317–396 (1951); these Rev. 13, 787] for the special case when spherical symmetry is assumed. A form for the five-dimensional stress energy tensor is assumed and the results are compared to the equations satisfied by the gravitational field of a charged sphere in general relativity. This is also done for the region where the stress energy tensor vanishes.

A. H. Taub (Urbana, Ill.).

Johnson, C. Peter, Jr. A criticism of a recent unified field theory. Physical Rev. (2) 89, 320–321 (1953).

In this note the author points out that Einstein's generalized field theory and indeed any unified field theory in which the gravitational potentials and the electromagnetic field strengths are defined by field equations which are numerically invariant under the transformation $x^{*\sigma}=kx^\sigma$, $\sigma=1, 2, 3, 4$, (i.e., which have the property that if a set of functions $g_\alpha(x)$ are solutions, so are $g_\alpha(kx)$) are in contradiction with the Newtonian and Coulomb law of force between charged masses. This is done by assuming two solutions of the field equations corresponding to two charged and one uncharged masses separated by distances large compared with their physical dimensions, all substantially at rest at time $t=0$. The two solutions are assumed to be related by the transformation given above. The charges and masses creating the fields, the accelerations they undergo and their mutual separations are compared for the two cases. It is then concluded that a contradiction exists with the Newtonian and Coulomb laws of force.

A. H. Taub.

Einstein, Albert. A comment on a criticism of unified field theory. Physical Rev. (2) 89, 321 (1953).

This note is Einstein's comment on the note discussed in the preceding review. Einstein points out that although the

field equations themselves have the homogeneity property used by Johnson they may not satisfy a further assumption made by the latter, namely, that two solutions of the type assumed by Johnson can coexist in the same world without destroying each other through their interactions. Einstein also gives two properties that field equations must satisfy in order that they be acceptable. One of these is essentially that Johnson's assumption is false. *A. H. Taub.*

Jaiswal, J. P. On the null geodesics and null cones in some gravitational fields. Ganita 2, 23–32 (1951).

Explicit expressions are obtained for the null geodesics and the null cone of a spatially flat expanding universe, and for the null geodesics of the Schwarzschild metric.

A. Schild (Pittsburgh, Pa.).

Raychaudhuri, Amal Kumar. Radiation sphere in Einstein universe. Bull. Calcutta Math. Soc. 44, 31–36 (1952).

The author studies the equilibrium of an Einstein universe with localized density inhomogeneities. He concludes that spherical pockets of black body radiation can exist.

A. Schild (Pittsburgh, Pa.).

Gilbert, C. Statistical systems of particles in the expanding universe. Quart. J. Math., Oxford Ser. (2) 3, 161–170 (1952).

A "kinematic system" is defined for the general space-time of relativistic cosmology. It consists of a system of fundamental observers together with a statistical system of material particles. In the limiting case when the pressure of matter is negligibly small the kinematic system determines a class of kinematic models of the universe. Included in this class of models are the Lemaître models of zero pressure, Milne's substratum, and Dirac's model. (From the author's summary.) *A. Schild* (Pittsburgh, Pa.).

Littlewood, D. E. Conformal transformations and kinematical relativity. Proc. Cambridge Philos. Soc. 49, 90–96 (1953).

Milne [Proc. Roy. Soc. London. Ser. A. 200, 219–234 (1950); these Rev. 11, 547] showed that the Einstein velocity-addition formulae are valid in a more general system than that of the Lorentz formulae. The object of the present paper is to investigate the most general system in which Einstein's formulae hold. The author considers a general Riemannian geometry (4-dimensional, signature ± 2) with local metric expressible in the Minkowski form and subjects it to an arbitrary conformal transformation, $g'_{ij}=e^{2\phi}g_{ij}$. He suggests that in place of Einstein's method in General Relativity, the conformal curvature be equated to zero and the total curvature be zero in the absence of matter. This leads to a new type of potential equation $\Box^2(e^{\phi})=0$ in place of the older $\Box^2(\phi)=0$, the difference being significant only on the cosmological scale. Applying these ideas to a uniformly expanding world-model, the author deduces that the modifying factor e^{ϕ} leads to a transformation resembling that connecting the t and τ scales of Milne's theory, i.e., $\tau \propto \log t$. To ensure convergence of gravitational potential at great distances, however, the author has to postulate either a tailing-off of density at large velocities or else an upper limit to the velocity of recession which, by comparison with observation, would appear to have to be between two-fifths and seven-eighths that of light. *G. J. Whitrow* (London).

MECHANICS

Steward, G. C. On certain configurations of the cardinal points in plane kinematics. Acta Math. 88, 371–383 (1952).

The set of cardinal points A_n in the fixed plane and the set A_n' in the moving plane are the instantaneous centers of rotation, of velocity, and of the successive orders of acceleration. If $\sigma_{n,r}=(-1)^{r+1}\binom{n}{r}$, and masses proportional to $\sigma_{n,r}$ are placed at A_r for $r=1, 2, 3, \cdots, n$, then A_n' coincides with their center of mass. Relations among the centrodes, the paths of the A_n and A_n', are derived analytically by the use of complex operators. As special cases, it is shown that if the cardinal points of a set lie on a straight line, alternately on two perpendicular straight lines, on a logarithmic spiral, or at the vertices of a spiral rectangular polygon (an orthogonal set), then these configurations are preserved during the motion. The motions associated with these configurations are discussed. A comparable treatment of these cardinal points is given by Bereis [Österreich. Ing.-Arch. 5, 246–266 (1951); 6, 246–255 (1952); these Rev. 13, 292; 14, 99]. *M. Goldberg* (Washington, D. C.).

Eremeev, N. V. On a nomographic mechanism. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk 1952, no. 3, 9–14 (1952). (Russian)

A cam whose polar equation is $r=f(\theta)+r_0$ drives a linear cam follower. If the cam rotates at uniform speed, the motion of the cam follower is a similar function of the time. Each value of r_0 will give another cam of a family which will drive the cam follower with the same motion. However γ, the inclination of the cam follower to the cam surface, varies with r_0; in fact, tan γ is proportional to r. In mechanical applications, it is important to avoid values of γ below specified values. A nomographic mechanism which gives these minimum values for the cam family is described.

M. Goldberg (Washington, D. C.).

Ziegler, Hans. Zum Begriff des konservativen Systems. Elemente der Math. 7, 121–129 (1952).

Clarification of basic ideas on work and energy in elementary mechanics, with a discussion of generalised gyroscopic forces in Lagrange's equations. *J. L. Synge.*

Bustamante, Enrique. Elementary particles at rest. Physical Rev. (2) 88, 1179–1181 (1952).

A non-relativistic theory of gravitation is proposed in which the Newtonian potential $\gamma m/r$ of a point mass is replaced by $2c^2 \log(1+\gamma m/2c^2 r)$. Application to the stability of atomic nuclei and to the perihelion rotation of Mercury are discussed. *A. Schild* (Pittsburgh, Pa.).

Dobronrabov, V. V. On the application of the method of nonholonomic coordinates to certain questions of the mechanics of continuous media. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk 1950, no. 9, 31–35 (1950). (Russian)

The author determines the vector fields satisfying the equation $a=k$ curl a, with k an arbitrary point function. The reviewer fails to see why it is pertinent to mention anholonomic coordinates in this connection. On the contrary, when the coordinates used are anholonomic, the

integration theory of the first-order partial equations used by the author does not apply. *A. W. Wundheiler.*

Dobronravov, V. V. On some questions of the mechanics of nonholonomic systems. Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 760–764 (1952). (Russian)

The author defends himself against the charge [Neĭmark and Fufaev, same journal 15, 642–648 (1951); these Rev. 13, 394] that his generalization of the Jacobi equation to anholonomic coordinates was based on improper use of the commutativity condition (A) $d\delta x^i = \delta dx^i$. His defense consists in arguing that the same relation was applied by other men, including the reviewer. The latter does not agree: the pertinent paper [Prace Mat.-Fiz. 38, 129–146 (1931)] contains a section proving that (A) is necessary and sufficient for holonomicity. *A. W. Wundheiler* (Chicago, Ill.).

Hydrodynamics, Aerodynamics, Acoustics

***Partington, J. R. An advanced treatise on physical chemistry. Vol. 1. Fundamental principles. The properties of gases. Vol. 2. The properties of liquids. Vol. 3. The properties of solids.** Longmans, Green and Co., London-New York-Toronto, 1949, 1951, 1952. xlii+943 pp., xliv+448 pp., liii+639 pp. $16.50, $10.00, $14.00.

Readers of these Reviews may be surprized to read this title here. It may serve as notice that among the competitors for the field of classical mechanics, now abandoned by virtually all physicists, the chemists are not the least active and competent. This work is evidence of a lifetime of thought and industry on every aspect of the mechanical and thermal behavior of matter.

The mathematical reader should not be deceived by the author's odd sections on elementary mathematics and numerous slighting remarks concerning the more fundamental parts of the subject, but should recall that responsibility for the woeful lack of understanding of mathematical methods and their usefulness and function in natural science, whose existence is all too strongly demonstrated in this otherwise profound work, must be shared at least in equal part by mathematicians themselves.

This book should be used by the mathematician in the same way that the author implies mathematics should be used by natural scientists: a storehouse of formulae and facts. As such, its value is unequalled. It is impossible to outline its contents, which is better revealed by an example. Suppose one wishes to know what the physicists and chemists regard as viscosity, how it is measured, what the theory of the measuring instruments is, what existing theories have predicted and from what assumptions, how well the results fit the data, and what empirical formulae fit it better. All this one finds assembled, discussed, and interrelated, together with hundreds of references to the original sources, in parts VII F (gases) and VIII E (liquids). In a word, a mathematician wishing a quick but sound education on the physical aspects of any problem concerning gases, liquids, or solids will find it here.

In the preface to vol. 1 the author defends himself for having written a "pseudo-Teutonic Handbuch": "Those who dig deeply in this mine of ready-made material for their slimmer and more attractive volumes mostly omit to say where they have been for their material, and, if they give references, usually reproduce those of their unnamed sources, all the inaccuracies being carefully copied. It is, however, a mistake to assume that a capacity to present a field of science as a whole, and not merely a small part in which the author himself has worked, is incompatible with originality. . . ." The three volumes contain over 45,000 references. To many of the topics the author has himself at some time made valuable contributions. In the preface to vol. 3 he shows some reassurance: "The method of presenting material without any reference to the nature of its development, although still favored in some quarters, has ceased to be acceptable almost overnight. . . . the treatment of a subject is more attractive and informative when it elucidates how particular concepts arose. This aspect will probably receive increasing consideration in books of the future, and the type of publication in which the bulk of the text is a translation from a "Handbuch" (which is not mentioned), although necessarily good, will perhaps be less esteemed." The author's own work is concrete evidence of this new trend in scientific treatises. He takes pains to let the original discoverers of major ideas or results speak for themselves. Despite the enormous scholarship and detail and the dryness of some of the topics, every page is not only lucid but fascinating reading, presented in a highly personal and incisive fashion. A scrupulous and understanding reader of the works of previous centuries, the author does not let his respect for the classics outweigh his fairness to more recent work, nor his thorough knowledge of the latest literature overbalance his sense of the magnitude of fundamental past advances. For example, on the experimental equations of state for gases he gives about 80 pp., beginning with Anaxagoras and on some topics citing papers of 1947 and 1948. The author is able to give a complete and entirely reliable account of the facts, while letting his own views plainly appear. These numerous critical judgments, which the reviewer finds to be always considered and often penetrating, but with which he can only rarely agree, transform this work from a "Handbuch" into a lasting original contribution to mechanics. *C. Truesdell.*

Taylor, G. I. Formation of a vortex ring by giving an impulse to a circular disk and then dissolving it away. J. Appl. Phys. 24, 104 (1953).

Butler, S. F. J. A note on Stokes's stream function for motion with a spherical boundary. Proc. Cambridge Philos. Soc. 49, 169–174 (1953).

In the case of axially symmetric motion the author proves the analogue of the sphere theorem of Weiss [same Proc. 40, 259–261 (1944); these Rev. 6, 191] for the Stokes stream function $\psi(r, \theta)$, r, θ, φ being spherical polar coordinates. The theorem for motion external to a sphere takes the form: If $\psi_0(r, \theta)$ is the Stokes stream function for an unbounded fluid in irrotational motion, all singularities of ψ_0 being external to $r=a$ and $\psi_0(0, \theta)=0$, then if a rigid sphere of radius a, center at the origin, is introduced, the stream function is $\psi_0(r, \theta)-ra^{-1}\psi_0(a^2r^{-1}, \theta)$. A corresponding theorem for motion internal to a sphere is given and several applications are worked out. The proof relies on the fact that if $D\psi(r, \theta)=0$, then $Dr\psi(a^2r^{-1}, \theta)=0$, where

$$D=r^2\partial^2/\partial r^2+\sin\theta(\partial/\partial\theta)[(1/\sin\theta)\partial/\partial\theta].$$

J. V. Wehausen (Providence, R. I.).

Pataraya, N. N. On the hydrodynamic interaction of spheres moving together in a fluid. Soobščeniya Akad. Nauk Gruzin. SSR. 11, 3–9 (1950). (Russian)

In experiments by V. V. Šuleĭkin [Fizika morya, Moscow-Leningrad, 1941, pp. 733–746] steel spheres projected line abreast from the stern of a ship and cine-photographed during their sinking were observed to draw apart as if mutually repelled. Quoting from his (unavailable) thesis [Tiflis University, 1948] the author states expressions derived partly theoretically and partly empirically for the hydrodynamic force on each of a system of n spheres moving line abreast. The component, in the line of centres, is for each sphere the sum of two parts, an attractive force proportional to the square of the velocity and a force of repulsion proportional to the acceleration, so that if the spheres start with sufficiently small velocity, the repulsive force should dominate the initial stages of the motion.

L. M. Milne-Thomson (Greenwich).

***Cooper, Eugene P. Use of conformal mapping in the study of flow phenomena at the free surface of an infinite sea.** Construction and applications of conformal maps. Proceedings of a symposium, pp. 87–89. National Bureau of Standards, Appl. Math. Ser., No. 18, U. S. Government Printing Office, Washington, D. C., 1952. $2.25.

Woronetz, Constantin. L'influence de la pesanteur sur la forme du jet liquide. C. R. Acad. Sci. Paris 236, 271–273 (1953).

Lewy, Hans. On steady free surface flow in a gravity field. Comm. Pure Appl. Math. 5, 413–414 (1952).

Dans un travail récent [Proc. Amer. Math. Soc. 3, 111–113 (1952); ces Rev. 14, 168] l'auteur a établi l'analyticité de la ligne libre, en dehors des points de vitesse nulle, dans le cas du mouvement à potentiel d'un fluide pesant dans un plan vertical. En utilisant les résultats de ce travail, l'auteur retrouve très simplement la propriété connue du mouvement de pouvoir être déterminé dans le voisinage de la ligne libre lorsque l'équation (analytique) de celle-ci est donnée. Les équations du mouvement sont explicitées sous forme paramétrique et leur domaine de validité dépend évidemment des singularités de la fonction analytique qui détermine la ligne libre.

R. Gerber (Toulon).

Ursell, F. Mass transport in gravity waves. Proc. Cambridge Philos. Soc. 49, 145–150 (1953).

"A proof is given of the existence of a slow drift associated with the passing of gravity waves over the surface of a frictionless fluid when the wave motion is irrotational. The drift is in the direction of propagation and is greatest near the surface, decreasing steadily towards the bottom, where it may be negative if the fluid is of finite depth. The particle displacement due to a solitary wave has similar properties." Let $\varphi(x, y)$ and $\psi(x, y)$ be the velocity potential and stream function in a coordinate system giving steady motion. The author expresses x and y as Fourier series (integrals in the case of the solitary wave) in φ and calculates $T(\psi_0)$, the time necessary for a particle on the streamline $\psi=\psi_0$ to move one wave length, as an expression in the Fourier coefficients. Suitable definition of the mass transport velocity and further computation lead to the result.

J. V. Wehausen (Providence, R. I.).

Binnie, A. M. The stability of the surface of a cavitation bubble. Proc. Cambridge Philos. Soc. 49, 151–155 (1953).

The author considers the stability of the interface of a hollow sphere of liquid of density ρ_1 surrounded by a spherical shell of liquid of density ρ_2 when the interface is accelerated from rest. The effect of surface tension is included. If surface tension is zero, the results conform with those of G. I. Taylor for a plane interface [Proc. Roy. Soc. London. Ser. A. 201, 192–196 (1950); these Rev. 12, 58].

J. V. Wehausen (Providence, R. I.).

Timman, R., et Lemaigre, B. La ligne portante de forme arbitraire considérée comme cas limite d'une surface portante en fluide incompressible. Nationaal Luchtvaartlaboratorium, Amsterdam. Report F.95, 19 pp. (1951).

The author presents a possible reduction of the lifting surface integral equation for uniform incompressible flow to a lifting line equation which, for the unswept wing, reduces to Prandtl's equation. *E. Reissner* (Cambridge, Mass.).

Golubev, V. V. Investigations on the theory of a flapping wing. Moskov. Gos. Univ. Učenye Zapiski 154, Mehanika 4, 3–53 (2 plates) (1951). (Russian)

This summarizes the author's work on flapping wings, partially reported in Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk 1946, 641–658, and other less accessible places. He has determined the average thrust X_0 and lift Y_0 per unit span during one period T of incompressible non-viscous plane flow about an airfoil of chord b moving forward with constant velocity V and periodically oscillating normal to V with amplitude h, frequency N, constant velocities $(-1)^j w$ $(j=1, 2)$, and constant angles of attack θ_j relative to V. Experiments performed in 1948 by Ya. E. Polonskiĭ with such a flapping wing in a water channel suggest the following theoretical model for the flow (also described, e.g., in Durand's Aerodynamic theory [Springer, Berlin, 1935, v. 2, div. E, ch. V]). Vortices of constant strength $(-1)^j\gamma$ are shed at the end of each half cycle and form a Kármán vortex street behind the airfoil of width h parallel to V, the signs of the vortex strengths being the opposite of those for a non-oscillating obstacle, however. If the transverse motion is interrupted at the end of a half cycle and continued with no change of velocity or angle of attack, then Kelvin's theorem implies that during this half cycle the circulation about the airfoil was

$$\Gamma_j=-\pi b(V^2+w^2)^{\frac{1}{2}} \sin\left(\tfrac{1}{2}\alpha+\theta_j-(-1)^j \arctan w/V\right),$$

where α depends on the airfoil's camber. Now

$$(1)\qquad \gamma=\Gamma_2-\Gamma_1=2\pi b \cos\sigma(V \sin\delta+w \cos\delta),$$

where $\sigma=\alpha+\frac{1}{2}(\theta_1+\theta_2)$, $\delta=\frac{1}{2}(\theta_1-\theta_2)$. By the theory of vortex streets, (2) $u_0=(\gamma/2l)\tanh(h\pi/l)$ where u_0 is the velocity of the vortices, directed oppositely to V, and l is the vortex spacing in either row. For flapping airfoils $l/2h=(V+u_0)/w$, which with (1), (2), and $b/h>0$ implies $w/V>2h/l$.

Let x, y be a rectangular coordinate system relative to which the street's vortices are stationary. Suppose that at some time, say $t=0$, these vortices are located at $\zeta_k=a+kl+\frac{1}{2}hi$ (strength γ) and $\zeta_k'=a+(\frac{1}{2}+k)l-\frac{1}{2}hi$ (strength $-\gamma$), where a, h, l are constants, and $k=0, 1, 2, \cdots$. Let S be the rectangle with boundary L and vertices $\pm H^{1\cdot 6}\pm Hi$, where $H^{1\cdot 6}=a+(k+0.75)l$ for some integer k is so large that the airfoil and new vortices that appear during $0\leq t\leq T$ lie in S. Consideration of the flow of mo-

mentum through S during a period T yields

$$(Y_0+iX_0)T/\rho=-\frac{1}{2}\int_0^T dt\int_L (dW/dz)^2dz$$
$$-\int_L [\varphi]_0^T(dx-idy)-i\left[\iint_S (dW/dz)dxdy\right]_0^T,$$

where $z=x+iy$, and $W(z,t)=\varphi+i\psi$ is the complex velocity potential. Since the only singularities of W outside of S are the street's vortices,

$$(3)\quad dW/dz-(\gamma/2\pi i)\sum_0^\infty[(z-\zeta_k)^{-1}-(z-\zeta_k')^{-1}] =-u_0+a_1/z+a_2/z^2+\cdots.$$

The circulation about L yields $4\pi i a_1=\Gamma_1+\Gamma_2-\gamma$, while knowledge of the forms of the coefficients $a_2, \cdots$, as functions of t is not required. By means of (3), $(Y_0+iX_0)T$ can be evaluated to within $O(H^{-1})$, and passage to the limit $H\to\infty$ yields

$$(4)\quad X_0/\rho=-\gamma^2\{(h\pi/l)\tanh(h\pi/l)-1\}/2\pi l-(V+u_0)\gamma h/l,$$

and $Y_0=-\frac{1}{2}\rho V(\Gamma_1+\Gamma_2)$. For $(h\pi/l)\tanh(h\pi/l)>\frac{1}{2}$ (or $h/l>0.245$) $X_0<0$, i.e., X_0 is a thrust. If the signs of γ and u_0 are reversed, (4) reduces to Kármán's equation for the drag of a non-oscillating obstacle. The condition for stability of vortex streets $\tanh(h\pi/l)=2^{-\frac{1}{2}}$ (or $h/l=0.281$) yields $w/V>0.562$ and

$$X_0=-0.94\rho b\cos\sigma(V\sin\delta+w\cos\delta)\cdot(V+1.2w).$$

For a two-dimensional flapping wing aircraft which moves so that its altitude is periodic with period T, a motion defined as level flight, the author draws the following conclusions. A necessary and sufficient condition for level flight is $\frac{1}{2}(P_1+P_2)=mg$, where $P_1(P_2)$ is the constant lift during the down-(up-)ward stroke, and mg the aircraft's weight. The mean altitude is attained at the end of each half cycle. Limitations imposed by the stalling angle of attack imply $\theta_1<0$ and $\theta_2>0$, which may account for the impression that birds in level flight seem to "row" through the air. The maximum attainable thrust

$$X_M=-(Y_{\max}-Y_0)2h/l=-0.562(Y_{\max}-Y_0),$$

where $Y_{\max}$ is the airfoil's maximum lift per unit span, and Y_0 the lift required for steady, non-flapping level flight. Finally, the author considers numerical examples which give reasonable agreement with observed values of V, h, and N for level flight of doves and flies, and which account for a certain unsuccessful ornithopter's failure to fly.

J. H. Giese (Havre de Grace, Md.).

Golubev, V. V. On some questions of the theory of a flapping wing. Moskov. Gos. Univ. Učenye Zapiski 152, Mehanika 3, 3–12 (1951). (Russian)

This treats the case $V=0$, $\alpha=0$ (symmetrical airfoil), $\theta_1=-\theta_2$ of the preceding paper for application to the hovering flight of birds in which the body is roughly vertical and the wings flap more or less horizontally. The spacing $l=h/0.281$ and velocity $u_0=\gamma/2^{1\cdot5}l$ of the vortex street and $N=u_0/l$ yield $h=w/2N=0.34b\cos\theta_1$. Also $X_0/b=P/S$ yields $N^2=P\lambda/(0.57\rho S^2\cos^3\theta_1)$, where P grams is the bird's weight, and S cm^2 and $\lambda=S/b^2$ are the wing's area and aspect ratio. For hummingbirds (sea gulls) an empirical formula $S^{1/2}P^{-1/3}=4.2$ and assumptions $1<\lambda<2$ ($\lambda=7.6$), and $30°<\theta_1+90°<45°$ yield computed frequencies from about two to five (one to three) times the observed values.

J. H. Giese (Havre de Grace, Md.).

Timman, R. The aerodynamic forces on an oscillating aerofoil between two parallel walls. Appl. Sci. Research A. 3, 31–57 (1951).

An exact solution, in infinite series form, but without numerical results is given for the pressure distribution on an oscillating airfoil halfway between two parallel walls, under the assumption of two-dimensional incompressible flow. The analysis is based upon the formulae for the field of a source or vortex in the presence of the airfoil and the walls which can be obtained by means of a conformal transformation of the field into a rectangle. The potential can then be expressed in terms of elliptic θ-functions.

It may be remarked that similar results including some numerical calculations may also be found in a doctoral dissertation by C. H. Gordon [Mass. Inst. Tech., ~~1952~~]. 1951
For numerical results for this problem by an approximate method, which were confirmed by Gordon's work, reference may also be made to a publication of the reviewer [Cornell Aeronaut. Lab. Rep. no. **SB-318-S-3** (1947)].

E. Reissner (Cambridge, Mass.).

Cabannes, Henri. Etude de quelques propriétés caractéristiques des solutions des équations de Navier. Bull. Soc. Math. France **80**, 37–46 (1952).

The author proves that the only three-dimensional flows of a viscous incompressible fluid which are uniform and regular at infinity (i.e., velocity expansible in negative powers of r for large r) must be irrotational and therefore potential flows. For plane and axially symmetric flows he establishes the same result even without the assumption of uniformity at infinity. These results are of interest in delimiting the class of viscous flows that may be considered in flow past a finite body. *D. Gilbarg.*

Yatseyev, V. I. On a class of exact solutions of the equations of motion of a viscous fluid. NACA Tech. Memo. no. **1349**, 7 pp. (1953).

Translated from Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **20**, 1031–1034 (1950); these Rev. **12**, 552.

Nigam, S. D. Advancement of fluid over an infinite plate. Bull. Calcutta Math. Soc. **43**, 149–152 (1951).

This paper deals with an unsteady flow of a viscous fluid over an infinite plate. The fluid initially is at rest and occupies an infinite space bounded by the plate $y=0$ and a front $x=0$. At time $t=0$, the front starts to move with a uniform velocity U_0 in the positive x-direction. The flow near the plate is solved by boundary-layer approximation. Both laminar and turbulent cases are considered. *Y. H. Kuo.*

Görtler, H. Zur laminaren Grenzschicht am schiebenden Zylinder. I. Arch. Math. **3**, 216–231 (1952).

Recent work, beginning with a note by Prandtl [Albert Betz Festschrift, Aerodynamische Versuchsanstalt, Göttingen, 1945, pp. 134–141], on the subject of the three-dimensional laminar boundary-layer flow on yawed infinite cylinders is reviewed briefly. Assuming the profile of chordwise flow components, $u(x,z)$ and $w(x,z)$, to be expressed in the so-called Blasius series, i.e., the potential-flow chordwise component $U(x)$ to be expressed in a power series, the author undertakes to compute the corresponding terms of the series for the spanwise velocity component $v(x,z)$. The investigation is limited to cases of symmetrical cylinders at zero incidence, for which $u(x,z)$ is odd in x, while $w(x,z)$ and $v(x,z)$ are even. The second-order ordinary differential equations satisfied by the first seven coefficients of the series

for $v(x, z)$ are written out; these seven coefficients are sufficient to express $v(x, z)$ to order x^8. They and their first derivatives are tabulated and graphed. *W. R. Sears.*

Shercliff, J. A. Steady motion of conducting fluids in pipes under transverse magnetic fields. Proc. Cambridge Philos. Soc. 49, 136–144 (1953).

It follows from the standard equations of hydromagnetics that the steady flow of an electrically conducting fluid in a pipe bounded by the planes $x=\pm a$ and $y=\pm b$ when there is a magnetic field of intensity H_0 acting in the z-direction is described by the equations

$$(1)\qquad \nabla^2 H_z+4\pi\sigma\mu H_0\partial v_z/\partial x=0$$

$$(2)\qquad \eta\nabla^2 v_z+(\mu H_0/4\pi)\partial H_z/\partial x+k\eta=0$$

where v_z and H_z are the components of the velocity and the magnetic field in the z-direction, η, σ, and μ are the coefficients of viscosity, electrical conductivity, and magnetic permeability respectively and k is a constant. In equations (1) and (2) ∇^2 is the two-dimensional Laplacian. Solutions of equations (1) and (2) are sought which lead to vanishing v_z and H_z on the walls. Letting $v=v_z\pm H_z/4\pi(\sigma\eta)^{1/2}$ we find that equations (1) and (2) can be combined in the form

$$\nabla^2 v\pm(M/a)\partial v/\partial x+k=0$$

where $M=\mu H_0 a(\sigma/\eta)^{1/2}$. The solution of this equation is obtained in the form

$$(3)\qquad v=\frac{16kb^2}{\pi^3}\sum\frac{(-1)^n}{(2n+1)^3}\left\{\frac{1+e^{m_1x}\sinh m_2a-e^{m_2x}\sinh m_1a}{\sinh(m_1-m_2)a}\right\}\times\cos\frac{(2n+1)\pi y}{2b},$$

where m_1 and m_2 are the roots of the equation

$$m^2\pm Mm/a-(2n+1)^2\pi^2/4b^2=0.$$

The behavior of the solution (3) for large M is such that one can say that a "boundary layer" occurs near the walls while in the core, the solution corresponds to "slug flow". The predictions regarding the mean velocity of flow derived from (3) are compared with the available experimental data and agreement is found. Some general remarks on flows in pipes of other cross sections are also made.

S. Chandrasekhar (Williams Bay, Wis.).

Michael, D. H. Stability of plane parallel flows of electrically conducting fluids. Proc. Cambridge Philos. Soc. 49, 166–168 (1953).

In the theory of the stability of plane parallel flows in hydrodynamics it has been known [cf. H. B. Squire, Proc. Roy. Soc. London. Ser. A. 142, 621–628 (1933)] that if a velocity profile becomes unstable to a small three-dimensional disturbance at a given Reynolds number, then it will become unstable to a small two-dimensional disturbance at a lower Reynolds number. In this paper the author shows that this result continues to hold in the framework of hydromagnetics for plane parallel flows of electrically conducting fluids confined between two perfectly conducting planes when a uniform magnetic field is applied parallel to the flow.

S. Chandrasekhar (Williams Bay, Wis.).

Krzywoblocki, M. Z. E. On the equations of isotropic turbulence in magneto-hydrodynamics of compressible medium. Acta Physica Austriaca 6, 157–166 (1952).

The invariant theory of isotropic turbulence in an incompressible fluid and in the framework of magneto-hydrodynamics developed by the reviewer [Proc. Roy. Soc. London. Ser. A. 204, 435–449 (1951); these Rev. 14, 424] is extended to the case of a compressible fluid. Thus instead of defining $\overline{u_iu_j'}$ and $\overline{u_ih_j'}$, one defines $\overline{\rho u_i\rho' u_j'}$ and $\overline{\rho u_ih_j'}$ in the compressible case; for in this way the solenoidal character of the tensors is preserved. With this redefinition of the tensors, the theory proceeds as in the incompressible case with no essentially novel features. *S. Chandrasekhar.*

Krzywoblocki, M. Z. E. On the equations of the decay of isotropic turbulence in magneto-hydrodynamics. J. Phys. Sòc. Japan 7, 511–512 (1952).

Starting from the equation of turbulence given in an earlier paper [see the preceding review], the author derives the forms which these equations take when the distance between the points at which the physical quantities are measured is reduced to zero. In this manner the equations governing the dissipation of kinetic and magnetic energies are derived. *S. Chandrasekhar* (Williams Bay, Wis.).

Krzywoblocki, M. Z. V. On the equations of the decay of isotropic turbulence in compressible fluid. J. Phys. Soc. Japan 7, 299–300 (1952).

Starting from the equation governing the scalar defining the tensor $\overline{\rho u_i\rho' u_j'}$ [i.e., the equation which replaces for a compressible fluid the equation of von Kármán and Howarth for an incompressible fluid], the author derives the limiting form of the equation when the distance between the points at which the physical quantities are measured for correlation is reduced to zero. In particular, an equation for the time rate of change of $\overline{\rho u^2}$ is obtained. *S. Chandrasekhar.*

Krzywoblocki, M. Z. E. On the invariants in the turbulence in compressible viscous fluids. J. Franklin Inst. 254, 317–322 (1952).

The principal result established in this paper is that under conditions of isotropic turbulence, $\int_0^\infty\overline{\delta\rho\delta\rho'}r^2dr$ is an integral of the equation of continuity where $\overline{\delta\rho\delta\rho'}$ is the correlation in the fluctuations in the densities at two points separated by a distance r. [This result was established earlier by the reviewer, Proc. Roy. Soc. London. Ser. A. 210, 18–25 (1951); these Rev. 13, 596.] *S. Chandrasekhar.*

Proudman, I. The generation of noise by isotropic turbulence. Proc. Roy. Soc. London. Ser. A. 214, 119–132 (1952).

The emission of acoustic radiation by a finite region in isotropic turbulence is considered on the basis of Lighthill's recent developments [same Proc. 211, 564–587 (1952); these Rev. 13, 879]. On this theory the intensity of sound $I(x,t)$ at a large distance x from the turbulent region is given by

$$I(x,t)=\frac{\rho_0}{16\pi^2c^5x^2}\times\int\int\overline{\left[\frac{\partial^2}{\partial t^2}(u_x^2-\overline{u_x^2})\right]_{t-\xi/c}\left[\frac{\partial^2}{\partial t^2}(u_x'^2-\overline{u_x^2})\right]_{t-\xi'/c}}dydy'$$

where the quantities in the first bracket of the integrand refer to the point y at time $t-\xi/c$ while the quantities in the second bracket refer to the point y' at time $t-\xi'/c$ and $\xi=|x-y|$ and $\xi'=|x-y'|$; further, c denotes the velocity of sound and ρ_0 the mean density in the turbulent region. With some further approximations the author deduces from the foregoing formula that the acoustic power output, P, per

unit mass, is given by

$$P=\frac{1}{4\pi c^5}\int U\,dr$$

where

$$U=\overline{\left[\frac{\partial^2}{\partial t^2}(u_x^2-\overline{u_x^2})\right]\left[\frac{\partial^2}{\partial t^2}(u_x'^2-\overline{u_x^2})\right]}.$$

For isotropic turbulence the expression for U can be reduced to the form

$$U=12\left(\overline{\frac{\partial u_x}{\partial t}\frac{\partial u_x'}{\partial t}}\right)^2+4\overline{u_xu_x'}\,\overline{\frac{\partial^2 u_x}{\partial t^2}\frac{\partial^2 u_x'}{\partial t^2}}$$
$$+4\frac{\partial}{\partial t}(\overline{u_xu_x'})\frac{\partial}{\partial t}\left(\overline{\frac{\partial u_x}{\partial t}\frac{\partial u_x'}{\partial t}}\right)+\left[\frac{\partial^2}{\partial t^2}(\overline{u_xu_x'})\right]^2$$
$$-4\frac{\partial^2}{\partial t^2}(\overline{u_xu_x'})\overline{\frac{\partial u_x}{\partial t}\frac{\partial u_x'}{\partial t}}.$$

If f, ϕ and ψ are the defining scalars (in the sense of the theory of isotropic tensors [Robertson, Proc. Cambridge Philos. Soc. 36, 209–223 (1940); these Rev. 1, 286]) of the tensors $\overline{u_iu_j'}$, $\overline{\dot{u}_i\dot{u}_j'}$ and $\overline{\ddot{u}_i\ddot{u}_j'}$ (dots denoting differentiation with respect to t), the integral for P can be reduced still further to the form

$$P=\frac{2}{15c^5}\int_0^\infty\left[12\left\{\overline{\left(\frac{\partial u}{\partial t}\right)^2}\right\}^2\phi'^2+4\overline{u^2}\,\overline{\left(\frac{\partial^2 u}{\partial t^2}\right)^2}f'\psi'\right.$$
$$+4\frac{\partial}{\partial t}(\overline{u^2}f')\frac{\partial}{\partial t}\left\{\overline{\left(\frac{\partial u}{\partial t}\right)^2}\phi'\right\}+\left\{\frac{\partial^2}{\partial t^2}(\overline{u^2}f')\right\}^2$$
$$\left.-4\frac{\partial^2}{\partial t^2}(\overline{u^2}f')\overline{\left(\frac{\partial u}{\partial t}\right)^2}\phi'\right]r^4dr,$$

where primes denote differentiation with respect to r. The author next shows that if one assumes that the joint probability distribution of the turbulent velocities and their first derivatives at two points in space is normal, the scalars ϕ and ψ can be expressed in terms of f by using the equations of motion and continuity. In this way P is expressed as an integral over a function involving only f. The author finally evaluates the resulting integral for the case when f has the form given by Heisenberg's theory of turbulence [Proudman, ibid. 47, 158–176 (1951)]. In this manner the formula

$$P=-57\frac{d\overline{u^2}}{dt}\left(\frac{\overline{u^2}}{c^2}\right)^{3/2}$$

for the acoustic power output is obtained.

S. Chandrasekhar (Williams Bay, Wis.).

***Batchelor, G. K.** **The theory of homogeneous turbulence.** Cambridge Monographs on Mechanics and Applied Mathematics. Cambridge at the University Press, 1953. x+197 pp. $5.00.

Ce petit livre est une synthèse des résultats actuellement acquis au sujet de la turbulence homogène dans un fluide incompressible. Après un résumé historique, l'auteur définit (chap. II) l'outillage mathématique nécessaire à l'analyse d'un champ de vitesses aléatoires: lois de probabilités, moyennes stochastiques, fonctions spectrales. Dans le chapitre III (cinématique de la turbulence), il introduit l'incompressibilité et les conditions de symétrie, et il établit les formules classiques relatives aux corrélations doubles et triples en turbulence isotrope. Dans le chapitre IV, il étudie en détails certains problèmes, plus ou moins rapprochés de la réalité, où il est possible de linéariser les équations, et d'aller jusqu'au bout des calculs: oscillateur harmonique soumis à une force aléatoire, passage d'un courant turbulent à travers un tissus, effet d'une contraction brusque de la veine.

Au chapitre V, il applique les équations de Navier-Stokes à l'étude de l'amortissement de la turbulence: relations entre les corrélations doubles et triples de vitesse; équation d'évolution de l'énergie; application à la stabilité des grands tourbillons et à la structure de la turbulence dans son état final (prédominance de la viscosité-linéarisation-anisotropie de l'écoulement); équations des écoulements isotropes.

Ces équations ne suffisent pas pour calculer les corrélations doubles ou les fonctions spectrales en fonction du temps. Les chapitres VI et VII sont consacrés aux hypothèses physiques qui ont été proposées pour compléter les équations dynamiques. Hypothèse de l'indépendance des termes spectraux, aux grands nombres de Reynolds, et dans un domaine de nombres d'ondes suffisamment grands (p. 112). Lois dimensionnels de l'équilibre universel, d'après Kolmogoroff (p. 114). Etude des diverses tentatives proposées pour mettre ces hypothèses en équations (von Kármán, Heisenberg). Le chapitre VII, d'un caractère moins déductif que les précédants, traite de la décroissance de l'énergie turbulente, d'après les résultats expérimentaux. Il existe un état d'équilibre statistique, indépendant des conditions initiales, valable pour des tourbillons contenant environs 80% de l'énergie (hypothèse du quasi-équilibre, p. 150). Application à la décroissance de la fonction spectrale de la turbulence isotrope, d'après Heisenberg.

Dans le dernier chapitre, l'auteur compare avec une loi normale les lois de probabilité d'une composante u de la vitesse, et de ses dérivées spatiales $\partial^n u/\partial x^n$. Les écarts, négligeables pour u, deviennent importants lorsque u augmente. L'avant dernier paragraphe est consacré à l'étude de la covariance de la pression, les résultats étant explicités lorsque les corrélations quadruples de vitesse sont reliées aux corrélations doubles comme pour une loi normale. Dans le dernier paragraphe, quelques hypothèses sont émises sur la structure de la turbulence à petite échelle.

Une importante bibliographie termine l'ouvrage.

J. Bass (Berkeley, Calif.).

Liepmann, Hans W. **Aspects of the turbulence problem. Survey report.** Z. Angew. Math. Physik 3, 321–342, 407–426 (1952).

Ces deux articles constituent un résumé, souvent très condensé, de nos connaissances relatives à la turbulence et à ses effets sur certains phénomènes physiques. Après une courte introduction, l'auteur rappelle en six pages les propriétés utiles des fonctions aléatoires (stochastic functions) scalaires ou vectorielles, y compris les propriétés ergodiques (conditions d'équivalence entre les moyennes temporelles et les moyennes stochastiques) et une formule de Rice [Bell System Tech. J. 23, 282–332 (1944); 24, 46–156 (1945); ces Rev. 6, 89, 233] sur les zéros et les extrema des fonctions aléatoires.

Dans la III^e section sont rassemblés divers résultats, rigoureux ou approchés, concernant des problèmes régis par des équations fonctionnelles linéaires à second membres aléatoires. L'auteur en signale, avec plus ou moins de détails, un certain nombre d'exemples: théorie du fil chaud, effet de la turbulence sur la propagation des ondes (électromagné-

tiques ou acoustiques), mouvement d'un solide dans un fluide turbulent, modification du champ magnétique dans un fluide conducteur turbulent.

La IV^e section est un résumé des théories générales de la turbulence: forme générale des équations de base, y compris les termes de conductivité thermique, analogies cinétiques, indications sur les modèles simplifiés de turbulence, théories asymptotiques et dimensionnelles relatives au spectre et aux corrélations, existence de "structure turbulentes secondaires" à grande échelle, superposées au mouvement turbulent à petite échelle. La V^e section donne une vue d'ensemble sur la turbulence isotrope, y compris certains résultats récents sur les fluctuations de pression et de température.

Bibliographie à la fin de chaque article. *J. Bass.*

Graffi, Dario. Il teorema di unicità nella dinamica dei fluidi compressibili. J. Rational Mech. Anal. **2**, 99–106 (1953).

L'auteur établit un théorème d'unicité pour le problème aux limites posé par le mouvement non permanent d'un fluide compressible, barotrope et homogène dans un domaine D fini ou non; le fluide est supposé soit visqueux, soit parfait. Les conditions initiales et les conditions à la frontière sont les suivantes. A l'instant initial les valeurs de $\mathbf{V}$ et ρ sont données dans D. Si D est fini et si le fluide est visqueux, les valeurs de $\mathbf{V}$ sont données pour tout $t>0$ sur la surface σ qui limite le domaine D et en outre les valeurs de ρ sont données en tous les points de σ où la composante normale de la vitesse est dirigée vers l'intérieur de σ; si le fluide est parfait, les conditions à la frontière se réduisent à la donnée de la vitesse normale et on suppose qu'en tout point de σ la composante normale de la vitesse est ou bien nulle, ou bien dirigée vers l'extérieur de σ. Si le domaine D est infini, certaines hypothèses précises sont faites sur la manière dont $\mathbf{V}$ et ρ tendent vers leurs valeurs à l'infini. L'auteur suppose que la borne inférieure de ρ est positive et il fait diverses hypothèses de continuité et de différentiabilité; il n'utilise pas la relation de Stokes entre les deux coefficients de viscosité. Sous les hypothèses ci-dessus l'auteur démontre qu'il ne peut exister plus d'une solution "régulière" du problème aux limites considéré. La méthode utilisée a été appliquée autrefois par l'auteur à un autre problème [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **12**, 129–135 (1930)]. *R. Berker* (Istanbul).

Germain, Paul. Solutions élémentaires des équations régissant les écoulements des fluides compressibles. C. R. Acad. Sci. Paris **234**, 1248–1250 (1952).

Using the Fourier transform as generalized by L. Schwartz "Théorie des distributions" [t. 2, Hermann, Paris, 1951; these Rev. **12**, 833], the author obtains an integral representation of the fundamental solution of the equation $u_{ss}+k(s)u_{\theta\theta}=0$. The location of the characteristics and the behavior of the fundamental solution on the characteristics is easily obtained from the integral representation.

B. Friedman (New York, N. Y.).

Sauer, Robert. Unterschallströmungen um Profile bei quadratisch approximierter Adiabate. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1951**, 65–71 (1952).

It is well known that the equation for the stream function of a plane irrotational flow takes the form

$$\psi_{\omega\omega}+\psi_{\theta\theta}+N(\omega)\psi_\omega=0 \tag{1}$$

in the variables θ, ω of the modified hodograph plane, where θ is the flow direction and ω is a known function of the flow speed. A further transformation $\psi=H(\omega)\psi^*$ brings (1) into the form (2) $\psi^*_{\omega\omega}+\psi^*_{\theta\theta}+F(\omega)\psi^*=0$. The essence of the Chaplygin-Kármán-Tsien approximation is to reduce (1) to Laplace's equation by setting $N(\omega)=0$, this being equivalent to replacement of the exact pressure density relation $p=p(\rho)$ by the special relation $p=A+B/\rho$; by proper choice of the constants the latter straight line in the $(p, 1/\rho)$-plane can be made tangent to the former adiabatic at any preassigned point. In the present paper the author observes that setting $F(\omega)=0$ in (2) again yields Laplace's equation, and that the special pressure-density curve for which this holds can be chosen to make quadratic contact with the true adiabatic. He therefore suggests that this approximation might be preferable to that of Kármán-Tsien in problems involving a larger density variation. The author considers analogously the pressure-density approximation (also quadratic) which reduces the equation for the Legendre transformation of the potential to Laplace's equation. The reductions to Laplace's equation considered in this paper and by Chaplygin and Kármán-Tsien are special cases of a general transformation theory of the equations of gas dynamics studied by Loewner [NACA Tech. Note no. **2065** (1950); these Rev. **13**, 464; Appl. Math. and Statist. Lab., Stanford Univ., Tech. Rep. No. **7** (1952)]. *D. Gilbarg.*

★Handbook of supersonic aerodynamics. Vol. 4. Section 12. Navord Report 1488. U. S. Government Printing Office, Washington, D. C. **$1.25.**

This section of the Applied Physics Laboratory handbook deals with aeroelastic phenomena in the supersonic regime, and is the work principally of T. K. Riggs and C. N. Warfield. The aerodynamic derivatives used are taken without comment from the linearized two-dimensional theory of Borbely [Z. Angew. Math. Mech. **22**, 190–205 (1942); these Rev. **5**, 136] and no experiments are quoted to justify this step. Brief reference is made to corrections for three-dimensional effects, but there is no mention of the grave doubts which have been cast on the adequacy of the linearized theory for normal thickness-chord ratios, and in particular on its prediction of negative damping for pitching oscillations about a mid-chord axis at low supersonic speeds, by experiments such as those of Bratt [Ministry of Supply [London], Aeronaut. Res. Council, Rep. no. **10709** (1947)] and theoretical work such as that of W. P. Jones and S. W. Skan [ibid., Rep. and Memoranda no. **2749** (in the press, see also reports with limited circulation **13162** and **14217**)].

The remainder of the work assumes these values for the aerodynamic derivatives and works out the critical flutter condition for two-dimensional torsional flutter and gives abbreviated accounts for two- and three-dimensional linay and tenay flexure-torsion flutter and two-dimensional tenay flexure-torsion-aileron flutter. *M. J. Lighthill.*

Chester, W. The decay of shock waves. Quart. J. Mech. Appl. Math. **5**, 408–422 (1952).

The first problem considered is the interaction of a shock with two centered expansion waves, initiated at different times by the motion of a piston in a gas-filled tube. By neglecting the third order quantities, the state behind the shock is taken to be uniform after it intersects the first simple waves. The shock path can then be obtained by integration. When the shock is hit by both waves, the states on both sides of the shock will be non-uniform. But, to the second order, the simple waves are not influenced by the shock, and its path can again be integrated. It is shown that

asymptotically shock decays inversely as the square root of the distance from the piston if it interacts with one expansion wave; and inversely as the distance from the piston if both expansion waves are present. Similar problems for the steady two-dimensional flow are also treated briefly.
Y. H. Kuo (Ithaca, N. Y.).

Adams, Mac C., and Sears, W. R. Slender-body theory—review and extension. J. Aeronaut. Sci. **20**, 85–98 (1953).

The theory of flow past slender bodies, elongated in the direction of flight, which finds its foundations in the work of Munk [NACA Rep. no. **184** (1924)], Jones [ibid. no. **835** (1946); these Rev. 11, 698], von Kármán and Moore [Trans. A. S. M. E. **APM-54**, 303–310 (1932)], Ward [Quart. J. Mech. Appl. Math. **2**, 75–97 (1949); these Rev. **10**, 644] and others is carefully reviewed. Particular attention is paid to the undetermined (by the conditions at the body) function of the streamwise variable that appears when Laplace's equation is solved for the transverse flow. Following Ward's [loc. cit.] general determination of this function in supersonic flow, the authors find it for subsonic flow, thereby filling an important gap in the previous work. The role of second order terms in the calculation of the pressure on the body is discussed and an interesting comparison between theory (with and without second order terms) and experiment is given. The calculation of the lift force from the virtual momentum of the transverse flow, including the effects of a trailing vortex wake, is described. A lucid interpretation of Ward's result for the drag is given, again including the wake. In discussing the extension to unsteady flow, the authors counter the reviewer's criticism [these Rev. 13, 181] of the use of the two-dimensional wave equation (rather than Laplace's equation) by Lomax, Heaslet and Fuller [NACA Tech. Note no. **2387** (1951)] in studying high frequency, unsteady flow past slender bodies and conclude (and the reviewer concurs) that Lomax et al. were justified in using the wave equation despite the fact that their boundary conditions were incorrectly formulated. The authors then proceed to extend slender body theory by expanding in powers of a width parameter ($|M^2-1|^{1/2}\times$span/chord) the kernels of the integral equations for symmetric and antisymmetric flow past thin wings in either subsonic or supersonic flight and obtaining approximate solutions by iteration. Their method is similar to that applied by Stewartson [Proc. Cambridge Philos. Soc. **46**, 307–315 (1950); these Rev. **11**, 699] to the special case of a rectangular wing in that the Laplace (Fourier for subsonic flow) transformation is employed but differs significantly (see below) in inverting the transforms prior to the approximate solution of the integral equations, rather than after as in Stewartson's paper. The method is applied to a flat triangular wing and the approximation consisting of the zero'th and second order terms in the width parameter compared with the complete linearized and experimental results, the error being about 15% in the relatively extreme case where the leading edges of the wing coincide with the Mach cone from its apex. [This example, together with a brief description of their extension, was given previously by the authors in J. Aeronaut. Sci. **19**, 424–425 (1952).] It is remarked that certain difficulties are experienced in the iteration procedure when the wing planform exhibits discontinuities in shape, and a technique for overcoming these difficulties is suggested and applied to an example. [The reviewer offers the following elaboration of a footnote by the authors. In inverting the integral equation it is implicitly assumed that the transform of the kernel has no zeros, corresponding to poles in the transform of the pressure distribution, that, if present, would lead to terms of order exp (−constant/width parameter) in the final results. Although in principle negligible compared with terms of algebraic order, these terms are found to furnish the most important (numerically speaking) correction to the zero order (Jones) result for a rectangular wing [cf. Stewartson, loc. cit.; Miles, J. Aeronaut. Sci. **18**, 770–771 (1951)]. It appears that such terms may be expected for any wing that exhibits a discontinuity in rate of change of span. It is not clear that the procedure suggested by the authors really overcomes this difficulty, and it would be of interest to recompute their example after separating out the poles in the Laplace transform inversion of the integral equation.] The paper concludes with a discussion of the application of the authors' extended slender wing theory to the determination of wing planforms of minimum drag. *J. W. Miles.*

Adams, Mac C., and Sears, W. R. On an extension of slender-wing theory. J. Aeronaut. Sci. **19**, 424–425 (1952).

Preliminary announcement of the paper reviewed above.

Flax, A. H. Reverse-flow and variational theorems for lifting surfaces in nonstationary compressible flow. J. Aeronaut. Sci. **20**, 120–126 (1953).

On the basis of linear theory, the author extends the reverse-flow theorem [same J. **19**, 352–353, 361–374 (1952); these Rev. **14**, 219,218] to the case of compressible nonsteady flow. Namely, if π and w denote respectively pressure divided by free-stream density and normal velocity and $\bar{\pi}$ and $\bar{w}$, the corresponding quantities in a reversed flow, then

$$\iint \bar{\pi} w dS = \iint \pi \bar{w} dS$$

where the integral is taken over the wing planform. This result is valid for the wing either oscillating harmonically or executing arbitrary time-dependent motion. By means of this theorem, a variational theorem is also established for this type of problem. Finally, several applications of the new reverse-flow theorem are given. *Y. H. Kuo.*

Zvereva, K. D. Determination of the moment of the force exerted upon a wing by a mass of compressible fluid in the case of plane steady flow. Moskov. Gos. Univ. Učenye Zapiski 152, Mehanika 3, 187–201 (1951). (Russian)

The author approximates, as follows, the moment on an airfoil $g(x, y)=0$ mounted at an angle of attack θ in a stream with low subsonic velocity W_∞ at infinity. First, by means of the hodograph equations $\partial s'/\partial q=\partial\vartheta'/\partial p$, $\partial s'/\partial p=-\partial\vartheta'/\partial q$, $\partial\varphi'/\partial p=K^{1/2}\partial\psi'/\partial q$, $\partial\varphi'/\partial q=-K^{1/2}\partial\psi'/\partial p$ he seeks in the $\xi=p+iq$ plane the compressible flow with circulation about $|\xi|=1$. Here φ' and ψ' are the potential and stream functions; $\lambda\cos\vartheta'$, $\lambda\sin\vartheta'$ is the velocity in units of the critical speed of sound; $s'=\int\lambda^{-1}(1-\lambda^2)^{1/2}[1-\lambda^2(\gamma-1)/(\gamma+1)]^{-1/2}d\lambda$; and $K^{1/2}=(\rho_0/\rho)(1-M^2)^{1/2}$. Let $\xi=\kappa(z)$ define a conformal mapping of the ξ-plane onto the $z=\mu+i\nu$ plane which takes $|\xi|=1$ onto $g(\mu,\nu)=0$, while $d\xi/dz=b(1+a_1\xi^{-2}+\cdots)$. The map from the z-plane onto the (x, y)-plane defined by $d\varphi=\lambda(\cos\vartheta dx+\sin\vartheta dy)$, $d\psi=\rho\lambda(\cos\vartheta dy-\sin\vartheta dx)$ yields the flow about a distorted version of $g(x, y)=0$. Then the moment (i) $L=\int_D[px-v\rho(uy-vx)]dx+\int_D[py+u\rho(uy-vx)]dx$,

where D is chosen as the image of $|\xi|=R$. To evaluate L explicitly as a function of $1-M_\infty^2$, θ, a_1, b, and the circulation $W_\infty V/b$ in the incompressible flow about $g(x,y)=0$, the author takes $\exp(s+i\vartheta)=W_\infty b^{-1}[e^{-i\theta}\xi+e^{i\theta}/\xi+(V\ln\xi)/2\pi i]$, determines expansions of φ' and ψ' about the point at infinity to terms of order $1/\xi$ or $1/\bar{\xi}$, and takes the limit of (i) as $R\to\infty$. *J. H. Giese* (Havre de Grace, Md.).

Couchet, G. Efforts aérodynamiques sur un profil animé d'un mouvement quelconque dans un fluide en repos à l'infini. O.N.E.R.A. Publ. no. 56, 32 pp. (1952).

The same author's previous work [same Publ. no. 31 (1949); these Rev. 12, 214] on profiles in arbitrary motion with constant circulation is extended here to the case where there are free vortices present in the fluid. Expressions for the force and moment on the profile are derived. The author states that his results agree with earlier ones of Bickley [presumably Philos. Trans. Roy. Soc. London. Ser. A. 228, 235–274 (1929)]. The locus of points is found at which a free vortex will stay fixed relative to a profile in uniform translation. In the final chapter the author proposes to construct the arbitrary motion of an airfoil profile, for which the Kutta-Joukowski condition is satisfied, as the limit of a succession of small motions with constant circulations. The resulting formulas do not look promising for practical calculation. No reference is made to work on a similar subject by R. Morris [Proc. Roy. Soc. London. Ser. A. 161, 406–419 (1937); 164, 346–368 (1938); 172, 213–230 (1939); 188, 439–463 (1947); these Rev. 1, 90; 8, 542] or E. E. Jones [Quart. J. Mech. Appl. Math. 4, 64–77 (1951); these Rev. 13, 175]. *W. R. Sears* (Ithaca, N. Y.).

Schuh, H. On asymptotic solutions for the heat transfer at varying wall temperatures in a laminar boundary layer with Hartree's velocity profiles. J. Aeronaut. Sci. 20, 146–147 (1953).

Lees, Lester. On the boundary-layer equations in hypersonic flow and their approximate solutions. J. Aeronaut. Sci. 20, 143–145 (1953).

Stenzel, Heinrich. Die akustische Strahlung der rechteckigen Kolbenmembran. Acustica 2, 263–264 (1952).

General formulae are obtained for calculating the sound field immediately in front of a vibrating rectangular membrane. The vicinal sound field is calculated by graphical integration and illustrated by many diagrams. Then general formulae for the radiation impedance are derived; and finally its two components are calculated and graphically constructed for different ratios of the sides of the rectangle ($a:b=1:1$, $a:b=2:1$, $a:b=5:1$, $a:b=10:1$). *Author's summary.*

Faran, James J., Jr. Scattering of cylindrical waves by a cylinder. J. Acoust. Soc. Amer. 25, 155–156 (1953).

Elasticity, Plasticity

✶Milne Thomson, L. M. Plane elastic problems. Conferencias de Matematica, Vol. III. Instituto de Matemáticas "Jorge Juan," Madrid, 1952. 27 pp.

Same as Revista Mat. Hisp.-Amer. (4) 9, 110–123, 141–153 (1949); these Rev. 11, 700.

Fridman, M. M. Solution of the general problem of bending of a thin isotropic elastic plate supported along an edge. Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 429–436 (1952). (Russian)

The plate described in the title is a multiply connected region with an outside closed boundary and k holes inside. The outside boundary is supported on hinges, on the boundaries of the holes are applied vertical loads through hinged washers, so that the bending moments at those boundaries are zero. The region of the whole plate is also loaded by a distributed load q. The author shows an analogy between his problem and the mixed problem in the plane theory of elasticity when the normal components of the displacements and the shearing components of the outside forces on the boundary are given. Utilizing this analogy he applies the complex functions used by D. I. Sherman [same journal 7, 413–420 (1943); these Rev. 6, 195], reduces the partial differential equation controlling the problem to a system of Fredholm integral equations, and proves that this system has a solution for every point of the region. *T. Leser* (Lexington, Ky.).

Mihlin, S. G. Estimate of the error in the computation of an elastic shell as a flat plate. Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 399–418 (1952). (Russian)

If an open shell and a plate, both of the same thickness and the contour of the plate being the greatest projection of the contour of the shell, are identically loaded and supported, then the difference of their potential energies will depend upon the curvatures of the shell; the smaller the curvatures the smaller the difference. This difference, divided by the energy of the plate, the author calls the error in the computation of a shell as a plate. He considers only shells of small curvatures and most of his paper deals with the derivations of convenient formulas and estimates of the energies of shells and plates. He presents an example of a helical shell where the error is less than 18%. The estimate of the error gives a better insight to the theory of shells of small curvatures, which the Russians call "sloping shells". The definition of a "sloping shell" is as follows: Given an element of an arc of a shell $ds^2=A_1^2(a_1,a_2)da_1^2+A_2^2(a_1,a_2)da_2^2$ and an element of an arc of a plate

$$ds_0^2=B_1^2(a_1,a_2)da_1^2+B_2^2(a_1,a_2)da_2^2$$

where a_1 and a_2 are curvilinear coordinates, if $|A_i-B_i|$ and $|\partial A_i/\partial a_k-\partial B_i/\partial a_k|$ are sufficiently small as compared with the smallest values of B_1 and B_2, then the shell is said to be a sloping shell.

The new insight is used by the author to present a new theory of sloping shells which differs from the existing theories of Vlasov [General theory of shells and its applications in technology, Moscow-Leningrad, 1949; these Rev. 11, 627] and Nazarov [Akad. Nauk SSSR. Prikl. Mat. Meh. 13, 547–550 (1949); these Rev. 11, 486]. The author points out also an error in Nazarov's work. *T. Leser.*

Nowacki, W. Plaque en bande orthotrope. Arch. Méc. Appl., Gdańsk 3, 259–270 (1951). (Polish. French summary)

L'auteur étudie la forme la plus convenable de la fibre moyenne de la voûte de pont en introduisant dans les calculs les paramètres α, β, γ, dont les valeurs on peut déterminer d'après la condition que le moment d'encastrement de la voûte doit être minimum. *Author's summary.*

Nowacki, W. Plaques rectangulaires avec les conditions aux limites mixtes. Arch. Méc. Appl., Gdańsk 3, 419–435 (1951). (Polish. French summary)

Dans le présent mémoire l'auteur donne la résolution précise de deux types de problèmes de la théorie des plaques minces, à voir: 1) Plaques rectangulaires à conditions aux limites différentes dans l'étendue de l'une des arêtes de la plaque (par exemple, un segment de l'arête est rigidement encastré tandis qu'un autre repose librement); 2) plaques rectangulaires supportées d'une manière quelconque sur leur périmètre et reposant en outre sur des appuis linéaires dans les limites de la plaque. Les problèmes de ce genre ont cette particularité que les conditions aux limites peuvent y être traduites par les équations intégrales de Fredholm de première espèce. Le procédé de la résolution du dit problème y est expliqué sur un exemple tout simple d'une plaque en demi-bande. *Author's summary.*

Szelągowski, Fr. Action d'une force concentrée sur la plaque indéfinie à l'aide d'une barre rigide de section circulaire. Arch. Méc. Appl., Gdańsk 3, 99–105 (1951). (Polish. French summary)

Ce problème détermine en point quelconque d'une tôle les valeurs des contraintes, dues à l'action d'une force P sur une partie centrale (une barre) rigide de section circulaire, assemblée avec cette tôle par soudure. La plus grande contrainte de traction (compression) et de cisaillement se produit sur le contour circulaire de la partie centrale rigide, et a pour valeur $P/2\pi R$. *Author's summary.*

Girkmann, K., und Tungl, E. Zum Anschluss von Stäben mit Winkelquerschnitt. Österreich. Ing.-Arch. 6, 255–265 (1952).

The angle is supposed attached at each end by one or two rivets or bolts on the centerline of its leg. Stresses in the attached and outstanding leg are computed by using infinite integral representations of Airy stress functions, one of which represents the stresses in an infinite sheet under a point load in the plane. Boundary and junction conditions are satisfied except at the free end of the angle. Comparison is made with the simple engineering solution. *D. C. Drucker* (Providence, R. I.).

Sáenz, A. W. Uniformly moving dislocations in anisotropic media. J. Rational Mech. Anal. 2, 83–98 (1953).

Dynamische Dislokationen in isotropen elastischen Medien wurden schon öfters untersucht; Ziel der vorliegenden Arbeit ist die auf den Fall von gleichmässig bewegten Dislokationen in anisotropen Medien zu erweitern. Diese diskontinuierlichen elastischen Felder sind eine Verallgemeinerung der zuerst von V. Volterra [Ann. Sci. Ecole Norm. Sup. (3) 24, 401–517 (1907)] untersuchten statischen Dislokationen.

Als wesentlich statische Dislokation wird eine solche bezeichnet, deren Lagenänderungsvektor u_i nur eine Funktion der x_i ist, wobei (1) $x_i = X_i - v_i t$ ist und die X_i die raumfesten Koordinaten und die v_i die Geschwindigkeitskomponenten der Dislokation bedeuten. Aus den bekannten Gleichungen der Elastizitätstheorie $\partial\tau_{ij}/\partial X_j = \rho\partial^2 u_i/\partial t^2$ und $\tau_{ij} = c_{ijkl}\epsilon_{kl}$, wo die τ_{ij} die Komponenten des Spannungstensors, die ϵ_{kl} die des Deformationstensors, und ρ die Dichte bedeuten und aus $c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}$ folgt

$$(2) \qquad (c_{ijkl} - \rho v_j v_l \delta_{ik}) u_{k,jl}.$$

Die Frage der Elliptizität von (2) wird besprochen. Für isotrope Medien ist das tatsächlich der Fall für $v < a$, wo a die transversale Schallgeschwindigkeit bedeutet.

Im dritten Teil der Arbeit wird dann die Integralformel von Somigliana auf wesentlich statische Lagenänderungen ausgedehnt und im vierten werden spezielle Lagenänderungsfelder konstruiert. Im fünften Teil werden die gleichmässig bewegten Volterraschen Dislokationen besprochen und im sechsten definiert der Verfasser die Somiglianaschen Dislokationen und gibt der Hoffnung Ausdruck, dass die Bewegung derselben durch Verallgemeinerung der hier mitgeteilten Berechnungen theoretisch bald lösbar sein wird. In der ganzen Arbeit wird die klassische lineare Elastizitätstheorie benutzt. *Th. Neugebauer* (Budapest).

Wittmeyer, H. Ein einfaches Verfahren zur näherungsweisen Berechnung sämtlicher Torsionseigenfrequenzen eines Stabes veränderlichen Querschnitts. Ing.-Arch. 20, 331–336 (1952).

The eigenvalue problem of the torsional vibrations of a shaft with smoothly varying cross section is treated by a parameter method which not only closely approximates the lower frequencies but also has the proper asymptotic values at very high frequencies. By a suitable modification of the equivalent variation problem, a perturbation parameter is obtained which admits the determination of the lowest frequencies to within a relatively small error. The method is readily adapted to other vibration and buckling problems for their eigenvalues but not the corresponding eigenfunctions. *D. L. Holl* (Ames, Iowa).

Duffin, R. J. Nodal lines of a vibrating plate. J. Math. Physics 31, 294–299 (1953).

Let $w(x, y)$ satisfy the differential equation $\nabla^2\nabla^2 w = k^2 w$ in the semi-infinite strip $0 \leqq x < \infty$, $-1 \leqq y \leqq 1$, and the boundary condition $w \to 0$, $\partial w/\partial y \to 0$ as $y \to \pm 1$. We also assume that all derivatives of w occurring in the differential equation are continuous and $O(e^{cx})$ uniformly, $c > 0$. For $0 < k < 4.635$ the function w must change its sign. The constant here is the best possible. Replacing the infinite region by a finite one, various consequences of the absence of nodal lines of w are listed. *G. Szegö.*

Das Gupta, Sushil Chandra. Transverse vibration of a wooden plate. Bull. Calcutta Math. Soc. 43, 143–146 (1951).

Following the procedure of Timoshenko [Philos. Mag. (6) 43, 125–131 (1922)] in the case of an isotropic beam for which the ratio of breadth to depth is either very small or very large, the author derives the frequency equation for the transverse vibrations of a beam of wood, an orthotropic material, in a state of plane stress. *H. W. March.*

Raher, W. Allgemeine Stabilitätsbedingung für krumme Stäbe. Österreich. Ing.-Arch. 6, 236–246 (1952).

The author reduces the problem of the instability of initially curved thin rods to an explicit form by giving a general formula for the second variation of the strain energy. Use is made of quasi-coordinates and the result obtained represents a generalization of earlier work by P. Funk [Österreich. Ing.-Arch. 1, 2–14 (1946); these Rev. 8, 242] for the initially straight rod. *E. Reissner.*

Rutecki, J. Instability of thin-walled bars with open cross-sections, their profile deformations being taken into account. Arch. Méc. Appl., Gdańsk 3, 437–460 (1951). (Polish. English summary)

The present paper contains an approximate solution of the problem of the stability of thin-walled bars with open cross-sections, the profile deformations being taken into

account. This solution has a general character and includes all possible profile forms, i.e., it enables the determination of critical lateral loads for any open cross-sections by means of selecting a suitable strain function.

From the author's summary.

Nowacki, W. De l'application du calcul des différences finies aux problèmes de la mécanique de construction. Arch. Méc. Appl., Gdańsk 3, 483–512 (1951). (Polish. French summary)

Ce mémoire traite de l'application du calcul des différences aux problèmes de la mécanique de construction concernant le flambage et la vibration des poutres et des plaques. La solution du système des équations linéaires qu'on obtient en substituant les coefficients différentiels aux dérivées dans les équations aux différences, y est amenée à l'expression propre à l'iteration. L'auteur envisage l'analogie advenant dans la résolution des dites équations et dans celle des équations intégrales, ainsi que des équations aux différences.

Author's summary.

Radok, J. R. M. Dynamic aero-elasticity of aircraft with swept wings. Coll. Aeronaut. Cranfield. Rep. no. 58, ii+42 pp. (2 plates) (1952).

This article gives general integro-differential equations covering the aero-elastic behaviour of an aircraft with sweptback wings. The kinetic and potential energy are calculated with respect to an oblique coordinate system taking into consideration the rigid body motions of the airplane and deformations of fuselage and wings. Hamilton's principle is used to find the equations of motion. The three types of problems: 1) free vibrations in vacuo, 2) flutter and dynamic stability, 3) gust loads, come under one heading. The equations of motion corresponding to these groups differ only by terms representing the external forces.

W. H. Muller (Amsterdam).

Geiringer, Hilda. Das allgemeine ebene Problem des ideal-plastischen isotropen Körpers. Österreich. Ing.-Arch. 6, 299–314 (1952).

The author considers the system consisting of the equilibrium relations, the continuity equation, a yield condition, and a set of stress-rate-of-strain relations for a perfectly plastic three-dimensional solid. In particular, it is assumed that these last relations need not be linear but are such that a linear relation exists between the stress-gradient of a plastic potential and the rate of strain tensor. The purpose of the author's work is to show that, for this case, a theory of plane stress or plane strain may be developed which depends upon two functions (the function specifying the yield condition and the function specifying the plastic potential), and which is a generalization of the usual theory of a perfectly plastic body. This is done by considering the characteristic equations of the system and the problem of approximate integration.

N. Coburn.

Csonka, P. Zur Theorie der plastischen Knickung. Acta Tech. Acad. Sci. Hungar. 5, 47–55 (1952). (Russian summary)

F. R. Shanley [J. Aeronaut. Sci. 13, 678 (1946); 14, 261–268 (1947)] has given a critical re-examination of column action in the plastic range. He conjectured that (1) bending will begin at the tangent-modulus load and (2) the maximum column load will be somewhere between the tangent- and reduced-modulus loads. Shanley then proved that these conjectures were true for a simplified column model (suggested by E. I. Ryder) consisting of a two-legged hinged column in which the hinge is a "unit cell" formed of two small axial elements, and in which the two legs are rigid. More detailed analyses for actual columns [e.g., by J. E. Duberg and T. W. Wilder, ibid. 17, 323–327 (1950); C. E. Pearson, ibid. 17, 417–424, 455 (1950); these Rev. 12, 143; and P. Cicala, ibid. 17, 508–512 (1950)] have confirmed qualitatively and also extended Shanley's original conjectures. The present paper also has the same object of extending Shanley's analysis and it has reference to constant H-section columns. However, the author's analysis is incorrect, and the author also appears unaware of the above work by Duberg and Wilder and by Cicala who discussed the action of similarly shaped columns. Cicala has shown that the exact solution of the problem depends upon the solution of an integro-differential system of equations, and he has devised a step-by-step method of procedure to yield approximate numerical results. Even when curvature of the stress-strain curve is neglected, the problem is still complicated, and this is due mainly to the fact that the column elements do not exhibit always the same mechanical behavior with changing column load: in two outer parts ("progression" zones) of the column both flanges undergo (compression) loading, but in the remaining inner part ("regression" zone) one flange undergoes (compression) loading and the other flange undergoes unloading (from compression); and, for example, if bending first occurs at the tangent modulus load, then the two boundaries between these parts move rapidly outwards from the mid-section towards the column ends with increasing load. Csonka's analysis (and therefore also the numerical results and conclusions based thereon) is incorrect because although such moving boundaries are found, the resulting complications in mechanical behavior of column elements are not considered; thus invalid relations between the stress and strain changes following column bending are employed.

H. G. Hopkins (Providence, R. I.).

Lee, E. H. A boundary value problem in the theory of plastic wave propagation. Quart. Appl. Math. 10, 335–346 (1953).

The author considers the one-dimensional propagation of an elastic or plastic wave in a cylinder, moving with constant velocity parallel to its axis and impinging against a rigid obstacle. On the end of the cylinder which impinges against the rigid obstacle, the stress builds up instantaneously from zero to the yield stress. As a result, an elastic wave is emitted which travels down the length of the cylindrical bar and is reflected as an unloading wave at the free end of the bar. Since the material adjacent to the impinging end becomes plastic, the elastic wave is followed by plastic waves (assumed to be simple waves). However, the reflected elastic wave unloads the stress at various plastic cross-sections until portions of the cylinder become elastic again. Due to the fact that this unloading wave is gradually absorbed, eventually a cross-section is reached where the elastic unloading wave no longer changes the plastic material into elastic material. At such a cross-section, either an elastic or a plastic wave may be propagated into the material. To determine what happens at such a cross-section, a particular example is discussed. The analysis is carried out by writing the propagation equations (due to G. I. Taylor and Th. von Kármán in the plastic state and H. F. Bohnenblust in the subsequent elastic state) in characteristic form and using numerical methods for the plastic region.

N. Coburn (Ann Arbor, Mich.).

Bishop, J. F. W. A theoretical examination of the plastic deformation of crystals by glide. Philos. Mag. (7) **44**, 51–64 (1953).

The macroscopic theory of plastic deformation in a polycrystalline metal is based upon observations of the behaviour of the metal in bulk, and upon the simplifying hypothesis that the material is locally homogeneous. The theory so constructed is found to be adequate as a first approximation when applied to many important problems in structural engineering and in metal forming processes. However, there is the important additional task of relating the macroscopic observations to ones more fundamental, and therefore, for example, to the mechanisms by which a single metal crystal deforms under stress. Much work has been directed towards this end, and the present paper describes developments following two previous papers by Bishop and Hill [Philos. Mag. (7) **42**, 414–427, 1298–1307 (1951); these Rev. **12**, 883; **13**, 603]. There it was assumed that glide (according to the Schmid law) was a sufficient mechanism for deformation of single metal crystals, and it was then proved that two extremum principles are valid. The extension of these principles to a polycrystalline aggregate led to the mathematically important result that the plastic potential and the yield function coincide for an aggregate (as they do for single crystals). This result in turn enables the yield function to be determined, and this was done for an isotropic aggregate of face-centered cubic crystals. In the present paper, there is given an investigation of the circumstances under which glide alone is, in fact, a sufficient mechanism for deformation of single crystals. It is proved that, if (at least) a single set of five independent shears exist amongst the available glide systems, any arbitrary strain (with zero hydrostatic part) can be imposed both geometrically and physically. The maximum work principle for a single metal crystal states that the actual stress corresponding to a given strain does not do less work than any other stress state satisfying the yield conditions. Here the yield surface $f=f(\sigma_{ij})$ is represented in an orthogonal cartesian space of six dimensions, the stress components σ_{ij} being plotted along the coordinate axes, and the above principle is employed to determine the yield surface of a single crystal of face-centered cubic metal, it being sufficient to determine the stress states defining the "vertices" of this surface. Differential hardening along slip directions is not considered, but this would cause changes in the size and shape of the yield surface. As an example of the use of the maximum work principle in conjunction with previously tabulated stress states, there is examination of the elongation of a single crystal of face-centered cubic metal with the transverse strains constrained to be equal. It is found that, although this particular strain may be imposed by only one stress state, there is available a triply-infinite set of shears.

H. G. Hopkins (Providence, R. I.).

MATHEMATICAL PHYSICS

Jordan, P. Über die Erhaltungssätze der Physik. II. Z. Naturforschung **7a**, 701–702 (1952).

[For part I see same Z. **7a**, 78–81 (1952); these Rev. **14**, 114.] The author discusses a relativistically invariant Lagrangean function which contains an electromagnetic vector potential, its first derivative, a meson potential and its first and second derivatives. This field theory is invariant under gauge transformations of both potential vectors.

A. H. Taub (Urbana, Ill.).

Ikeda, M. Note on invariancy of some fundamental equations of physics. Progress Theoret. Physics **8**, 382–383 (1952).

A review of the invariance of a number of basic physical equations (propagation of light in special relativity, $\Box\phi=0$, $\Box\phi^i=s^i$, Maxwell's equations, meson equations, Dirac's equations) relative to a number of transformations (Lorentz, translation, dilatation, an Abelian group with four parameters due to L. Page [Physical Rev. (2) **49**, 254–268 (1936)] connected with uniformly accelerated reference systems, and a transformation characterised by $\partial\xi^i/\partial x^i=$const. where ξ^i is the vector of infinitesimal transformation). The author concludes that it is almost impossible to enlarge the Lorentz group without deforming various equations which are used in the special relativistic frame.

J. L. Synge (Dublin).

Optics, Electromagnetic Theory

Satten, Robert A. An "algebra" of possibilities relating regions in object and image space for a system of thin lenses. J. Opt. Soc. Amer. **42**, 955–959 (1952).

The author develops a matrix algebra which permits coordination of the regions for which object and image are either real or virtual for a system of thin lenses with finite distances.

M. Herzberger (Rochester, N. Y.).

Lense, Josef. Über einen geometrischen Satz der Kristalloptik. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1951, 61–64 (1952).

The author gives an elementary proof of the existence of conical refraction in crystals.

M. Herzberger.

Durand, Emile. Détermination d'une trajectoire électronique par dérivations successives. C. R. Acad. Sci. Paris **236**, 471–473 (1953).

Iwata, Giiti. Realization of special contact transformations with static electromagnetic fields in vacuo. Progress Theoret. Physics **8**, 183–192 (1952).

The non-relativistic motion of a charged particle in a statical electromagnetic field is expressible in terms of a Hamiltonian $H(q,p)$ and the motion generates a contact transformation from an initial state (a,b) at time t_0 to a final state (q,p) at time t_1. The author considers coordinate-coordinate transformations in which the q's depend only on the a's, and finds some necessary conditions. He also finds necessary conditions for momentum-coordinate transformations in which the q's depend only on the b's (initial momenta). He is led to a Hamiltonian which is quadratic homogeneous in (q,p), and gives an example of an electromagnetic field generating a coordinate-coordinate transformation as well as a momentum-coordinate transformation. The application of these ideas to instruments involving charged particles is indicated in general terms. The coordinate-coordinate transformation, possible and realizable with static electromagnetic fields in vacuo, makes it possible to construct a perfect instrument by which all the particles emitted from a point source in any direction with any

velocity will be centered at another point, a stigmatic representation with no chromatic aberration. But, the magnification ratio of a perfect instrument being unity, the transformation will be applicable not to the design of an electron microscope but to the planning of a mass spectrometer.
J. L. Synge (Dublin).

Giovanardi, Ilde. Sulla propagazione delle onde elettromagnetiche in una guida riempita da un dielettrico eterogeneo. Ann. Univ. Ferrara. Sez. VII. (N.S.) 1, 81–87 (1952).

Chatterjee, S. K. Some perturbation effects in microwave cavities operating in degenerate modes. J. Indian Inst. Sci. 34, Sect. B, 77–87 (1952).

The author considers oscillations in a cylindrical cavity, operating in the degenerate $TE_{01n}-TM_{11n}$ modes, when the volume V of the cavity is slightly changed by introducing a metallic rod of volume v. For small values of v he assumes that the field inside the cavity is not altered by the presence of the rod, and evaluates the changes in the resonant frequencies of the two mode types because of the change in total volume. The change in the resonant Q values is also determined on the assumption that this is due solely to the power absorbed by the rod. The resulting formulas are somewhat complicated, since they involve the dimensions of the cavity and of the rod, but in a practical case the author finds that the change in resonant frequency for the TE_{01n} mode is much less than that for the corresponding TM mode, while the change in Q is much greater.
M. C. Gray (Murray Hill, N. J.).

Miller, M. A. Propagation of electromagnetic waves over a plane surface with anisotropic homogeneous boundary conditions. Doklady Akad. Nauk SSSR (N.S.) 87, 571–574 (1952). (Russian)

The author considers the propagation of electromagnetic waves over a plane surface $x=0$ defined electrically by the two surface impedances Z_1 and Z_2. By assuming that the field depends on time and the direction of propagation as $\exp[i(\omega t-hz)]$ and that the cartesian components of the field vectors satisfy the boundary conditions:

$$Z_1=\frac{E_z(0,y,z)}{H_y(0,y,z)},\quad Z_2=\frac{E_y(0,y,z)}{-H_z(0,y,z)},$$

he examines the phase velocity (ω/h) and the field configurations with Z_1 and Z_2 as parameters. *C. H. Papas.*

Dike, S. H. Difficulties with present solutions of the Hallén integral equation. Quart. Appl. Math. 10, 225–241 (1952).

This is a thoughtful contribution to the discussion of the various parameters that have been suggested for asymptotic solutions of Hallén's integral equation for the antenna current. The author considers the theoretical current distribution in a receiving antenna, and derives expressions for broadside absorption gain and back-scattering cross-section, which can be compared with experimental measurements. None of the proposed parameters leads to very satisfactory agreement with experiment, even with second order terms in the asymptotic expansion. Further, in the limiting case of very short antenna lengths, the gain and scattering cross-section reduce to the values for an elementary dipole only if the antenna is infinitely thin. The author suggests that a complex value of the expansion parameter might yield better results, or that a new approach to the problem should be attempted. *M. C. Gray* (Murray Hill, N. J.).

Ledinegg, E., und Urban, P. Zum ersten Randwertproblem der Maxwellschen Gleichungen. Ann. Physik (6) 10, 349–360 (1952).

The problem discussed by the authors is the existence of solutions of Maxwell's equations inside a closed volume V when the origin of the field is outside V and coupled to the interior through some parts of the surface. This problem arises, for instance, in the theory of electromagnetic cavity resonators when the cavity is excited from an external source. It is known that there is a unique solution of the wave equation $\Delta U+k^2U=0$ inside V, where U has assigned values on the surface F of V, as long as k is not one of the eigenvalues k_r of the homogeneous problem, where $U=0$ on F. But in the electromagnetic case such a solution does not in general satisfy the divergence condition on E. The principal theorem of the paper may be summarized as follows: There exists a unique solution, with continuous second derivative, of Maxwell's equations inside a closed volume V bounded by a surface F such that on F the tangential components of E may be assigned arbitrarily as a continuous differentiable field, as long as $k\neq k_r$.
M. C. Gray (Murray Hill, N. J.).

Roglić, Velimir. Une application de la condition de Saint-Venant en électrodynamique. Bull. Acad. Serbe Sci. (N.S.) 5, Cl. Sci. Math. Nat. Sci. Math. 1, 189–192 (1952).

The author wishes to demonstrate simply that Maxwell's electrostatic stress tensor cannot be identified with that of an elastic solid obeying Hooke's law [in the linear approximation]. His method is to show that an inverse square electrostatic field will not satisfy Beltrami's compatibility equations unless Poisson's ratio is 1. [It is well known that the electromagnetic field in vacuo is isomorphic in two ways to MacCullagh's quasi-elastic ether; cf., e.g., A. Sommerfeld, Vorlesungen über theoretische Physik, Bd. 2 [2nd ed., Dieterich, Wiesbaden, 1947, see §15; for a review of the 1st ed. see these Rev. 8, 356].] *C. Truesdell.*

***Stein, G. M. Conformal maps of electric and magnetic fields in transformers and similar apparatus.** Construction and applications of conformal maps. Proceedings of a symposium, pp. 31–57. National Bureau of Standards, Appl. Math. Ser., No. 18, U. S. Government Printing Office, Washington, D. C., 1952. $2.25.

***Mercier, André. Leçons sur les principes de l'électrodynamique classique.** Editions du Griffon, Neuchatel, Suisse, 1952. 74 pp. 7.80 Swiss francs.

A discussion of Maxwell's equations and the Poynting vector in which great attention is devoted to the dimensions of E, H, B, and D. A special notation is used to make clear that these quantities are, respectively, a polar vector, an axial vector density, an axial vector, a polar vector density. In a concluding section, Maxwell's theory is formulated axiomatically in terms of Clifford numbers.
A. J. Coleman (Toronto, Ont.).

Twiss, R. Q. Propagation in electron-ion streams. Physical Rev. (2) 88, 1392–1407 (1952).

Zur Berechnung der Ausbreitung von elektromagnetischen Wellen in Elektronen- und Ionenstrahlen (die aus N diskreten Teilstrahlen bestehen) wird eine mathematische

Theorie ausgearbeitet. Zur Lösung der Maxwellschen Differentialgleichungen wird zuerst ein System von drei aufeinander orthogonalen und zeitunabhängigen Grundvektoren

$$(1)\quad \mathbf{L}=\nabla\phi(x^1,x^2),\quad \mathbf{M}=\nabla\phi(x^1,x^2)\times\mathbf{a}\quad \text{und}\quad \mathbf{N}=\phi(x^1,x^2)\mathbf{a}$$

eingeführt. $\mathbf{a}$ bedeutet hier den in die z-Richtung zeigenden Einheitsvektor und x^1, x^2, und z bilden ein verallgemeinertes zylindrisches Koordinatensystem. ϕ ist eine skalare Funktion. Mit Hilfe von (1) wird E folgendermassen dargestellt:

$$(2)\qquad \mathbf{E}=E_1(z,t)\mathbf{L}+E_2(z,t)\mathbf{M}+E_3(z,t)\mathbf{N}.$$

Analoge Formeln gelten für $\mathbf{H}$ und $\mathbf{v}_e$. Es wird gezeigt, dass diese Gleichungen tatsächlich die Maxwellschen und Lorentzschen Gleichungen befriedigen, wenn

$$(3)\qquad \nabla^2\phi(x^1,x^2)+p^2\phi(x^1,x^2)=0$$

ist. Zu diesem Zwecke werden diese Ausdrücke in die entsprechend umgeformten Maxwellschen und Lorentzschen Differentialgleichungen eingesetzt und dann eine doppelte Laplacetransformation (bezüglich t und z) vorgenommen. Die schliesslich erhaltenen Resultate sind vom Typ

$$(4)\quad E_l(z,t)=\left(\frac{1}{2\pi}\right)^2\int_{-\infty-i\gamma_1}^{\infty-i\gamma}\exp(i\omega t)d\omega\int_{-\infty-i\gamma_2}^{\infty-i\gamma_2}\exp(ikz) \times\sum_{m=1}^{3}\left[C_m^*(k,\omega)+C_m^\dagger(k,\omega)\right]\frac{A_{ml}(k,\omega)}{\det\mathfrak{A}(k,\omega)}dk.$$

$\mathfrak{A}$ ist dabei eine dreireihige gewöhnliche Determinante und die C_m sind Elemente einer Kolonnenmatrix.

Unter gewissen einschränkenden Annahmen bezüglich der senkrechten Grenzbedingungen, kann bei willkürlichen Anfangsbedingungen die allgemeine Lösung mit Hilfe einer kompletten orthogonalen Reihe von elementaren Vektorlösungen dargestellt werden. Die richtige physikalische Deutung der bekannten "Ansatzlösungen" wird gefunden und einige spezielle Fälle und die Verallgemeinerung der Resultate auf kontinuierlich verteilte Geschwindigkeiten wird besprochen. Die Lösungen sind vollständig relativistisch.

Th. Neugebauer (Budapest).

Spitzer, Lyman, Jr. Equations of motion for an ideal plasma. Astrophys. J. 116, 299–316 (1952).

Formeln für die Bewegung eines Plasmas werden unter den Annahmen hergeleitet, dass erstens das Plasma vollständig ionisiert ist, zweitens dass die makroskopische Dichte und Geschwindigkeit sich während einer (im magnetischen Felde auftretende) Gyration und innerhalb der räumlichen Ausdehnung derselben kaum ändern, und drittens dass die freie Weglänge sehr gross im Verhältnis zum Gyrationsradius ist. Das tatsächliche Auftreten dieser Verhältnisse in kosmischen Problemen wird besprochen. Der Verfasser bezeichnet mit (n, m) die Annäherungen, die entsprechend der zweiten Annahme Glieder die klein von der n-ten Grössenordnung und entsprechend der dritten solche, die klein von der m-ten Ordnung sind, enthalten. Die Annäherung $(0, 0)$ ist trivial, die Grundgleichung für $(1, 0)$ wird aus der Arbeit von H. Alfvén [Cosmical electrodynamics, Oxford, 1950, Kap. II; diese Rev. 12, 756] entnommen. Die wird in der Form

$$(1)\quad \mathbf{V}_\perp=\frac{c\mathbf{B}\times\mathbf{F}}{ZeB^2}+\frac{mc}{ZeB^2}\left\{\tfrac{1}{2}w_\perp^2\mathbf{B}\times\nabla B+w_{11}^2\mathbf{B}\times\left(\mathbf{B}\cdot\nabla\frac{\mathbf{B}}{B}\right)\right\}$$

geschrieben, wo $\mathbf{V}_\perp$ die Geschwindigkeit eines Gyrationszentrums senkrecht zu B, w die Teilchengeschwindigkeit entlang des Larmorkreises, B die magnetische Feldintensität, und F die äussere Kraft bedeutet. Die übrigen Symbole haben die gewohnte Bedeutung. (1) bildet den Ausgangspunkt der ganzen Untersuchung. In der Näherung $(2, 0)$ erhält man die Polarisation, in der von $(3, 0)$ den Halleffekt und in $(2, 1)$ und $(3, 1)$ die Diffusionserscheinungen. Da der elektrische Strom senkrecht zu B grösstenteils vom Gradient des Druckes verursacht wird und das elektrische Feld eine gleichmässige Geschwindigkeit des Plasmas verursacht, so hängen das elektrische Feld und die Stromstärke nicht unmittelbar zusammen. Der Widerstand wird deshalb mit Hilfe der in Wärme umgewandelten Leistung definiert und auf dem Wege folgt, dass der Widerstand senkrecht zu B 1.96 mal so gross, als parallel zu B ist. Der Halleffekt, die transversale Viskosität und die Wärmeleitung sind sehr klein. *Th. Neugebauer* (Budapest).

Sakurai, Tokio. Microwave circuit theory. J. Phys. Soc. Japan 7, 185–189 (1952).

The author considers microwave solutions of Maxwell's equations for a closed surface which is perfectly conducting except over a number of isolated regions. By defining fictitious currents and voltages over these regions, in terms of the tangential electric and magnetic field components, it is possible to determine an impedance matrix Z_{pq} and a resonant quality factor Q analogous to the corresponding quantities in ordinary circuit theory. In particular, for the reactive case of zero conductivity on the subsurfaces, the quadratic form $\sum[\partial Z_{pq}/\partial(i\omega)]I_pI_q^*$ is positive definite.

M. C. Gray (Murray Hill, N. J.).

Roberts, T. E., Jr. Theory of the single-wire transmission line. J. Appl. Phys. 24, 57–67 (1953).

The author attempts to determine the current, radiation field, and admittance of a circularly cylindrical wire of infinite length fed by a coaxial line.

For the peripheral component of the magnetic field H_ϕ he deduces a contour integral, the integrand of which involves the electric field across the mouth of the coaxial line and the surface impedance of the wire. From this he obtains the far-zone H_ϕ (radiation field) by replacing the Hankel function of the integrand by its asymptotic value and then performing a saddle-point integration. This procedure has been used by A. A. Pistol'kors and others in the examination of closely related problems [Akad. Nauk SSSR. Žurnal Tehn. Fis. 17, 377–388 (1947)].

By integrating H_ϕ around the wire he obtains a contour integral representation for the current, although this method of determining the current is strictly valid only when the surface impedance of the wire vanishes. A deformation of this contour leads to a convenient dichotomy of the current, one part of which propagates along the wire and the other is attenuated with distance from the feed point. The former is due to the residue at a pole, the latter, to an infinite integral which he evaluates numerically. Graphs showing the current along the wire and the radiation field are presented.

He also considers the equivalent admittance $Y=G+iB$ terminating the coaxial line. For the sake of analytical simplicity he assumes the mouth of the coaxial line to be an infinitesimally thin circular gap and, therefore, only the conductance G remains finite. The susceptance B is infinite. To avoid such a "catastrophe" it is necessary to give the gap finite width, as was done by Levine and Papas for the circular diffraction antenna [J. Appl. Phys. 22, 29–43 (1951); these Rev. 12, 777]. *C. H. Papas* (Pasadena, Calif.).

★Fraenz, Kurt. **Mathematical problems of the theory of electric circuits of distributed constants.** Symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Diciembre, 1951, pp. 161–168. Centro de Cooperación Científica de la Unesco para América Latina, Montevideo, Uruguay, 1952. (Spanish)

Sternberg, R. L., and Kaufman, H. Applications of the theory of systems of differential equations to multiple non-uniform transmission lines. J. Math. Physics 31, 244–252 (1953).

Let $Z(x)$ and $Y(x)$ be matrices of series impedance and shunt admittance per unit length for a non-uniform transmission line of several conductors. Solutions of the differential equations for the voltages and currents on the multiple line are related to matrix solutions $H(x)$ of a matrix Riccati equation $H'-HZ(x)H+Y(x)=0$. For the case of a lossless multiple line, the non-existence of a solution with multiple voltage nulls at two points x_1, x_2 (condition for non-resonance of the line shorted at x_1 and x_2) is equivalent to the existence of a pure imaginary solution of the Riccati equation. *E. N. Gilbert* (Murray Hill, N. J.).

Cetlin, M. L. Application of matrix calculus to the synthesis of relay contact networks. Doklady Akad. Nauk SSSR (N.S.) 86, 525–528 (1952). (Russian)

Relay contact networks are considered which have the form of a box with p input terminals and p output terminals. A vector $\Lambda=(\lambda_1, \cdots, \lambda_p)$ is defined with coordinate $\lambda_i=1$ or 0 according as the ith input terminal is or is not at positive potential. The resulting condition of the output terminals is similarly given by a vector M. The operation of such a network is described by a vector equation $M=\Lambda A$ where A is a matrix of Boolean functions a_{ij} of the switching variables of the network ($a_{ij}=1$ when the network connects input terminal i to output terminal j and $a_{ij}=0$ otherwise). The matrix for two boxes connected in cascade is the product of the matrices of the separate boxes.

E. N. Gilbert (Murray Hill, N. J.).

Lunc, A. G. Algebraic methods of analysis and synthesis of relay contact networks. Izvestiya Akad. Nauk SSSR. Ser. Mat. 16, 405–436 (1952). (Russian)

An elaboration of the author's matrix and characteristic function methods for relay contact networks which were reported in Doklady Akad. Nauk SSSR (N.S.) 70, 421–423 (1950); 75, 201–204 (1950) [these Rev. 11, 574; 12, 779].

E. N. Gilbert (Murray Hill, N. J.).

Quantum Mechanics

Costa de Beauregard, Olivier. Sur le problème quantique et classique, de l'irréversibilité. C. R. Acad. Sci. Paris 236, 277–279 (1953).

Destouches-Aeschlimann, Florence. Dérivée et différentielle totale d'un opérateur. C. R. Acad. Sci. Paris 236, 354–355 (1953).

On définit les dérivées partielles, la différentielle totale, la dérivée totale par rapport au temps pour un opérateur d'une manière généralisant les règles usuelles de l'analyse. On montre que la dérivée totale est égale à la dérivée symbolique courante. De là résulte que toute intégrale première de la mécanique ondulatoire est intégrale première de la mécanique classique pour le problème correspondant.

Author's summary.

Fogel, Karl-Gustav. On the determination of eigenvalues and eigenphase for the Yukawa potential. Ark. Fys. 4, 573–579 (1952).

The Schrödinger equation for $l=0$ is solved by an expansion in terms of the binding constant. The eigenvalues appear as zeros of a real transcendental function of the energy parameter and the binding constant; the eigenphase, on the other hand, as the argument of a complex function of these variables. (From the author's summary.) *N. Levinson.*

Baumann, Kurt. Bericht über die neuere Entwicklung der Quantenelektrodynamik. I, II, III. Acta Physica Austriaca 5, 544–558; 6, 53–70, 195–208 (1952).

This is a series of expository articles, explaining the methods and results of the new quantum electrodynamics. The first paper is concerned with the basic ideas of Schwinger [Physical Rev. (2) 74, 1439–1461 (1948); these Rev. 10, 345], the second with the methods of removing the divergences from the theory of regularization and renormalization. In the third paper the theory is applied in detail to two standard problems, the scattering of two free electrons and the radiative correction (Lamb shift) of the energy-levels of a single bound electron. In the reviewer's opinion, this is the clearest and most useful introduction to the subject that has so far appeared. *F. J. Dyson.*

Rumer, Yu. B. Action as a space coordinate. VI. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. 22, 742–754 (1952). (Russian)

This paper continues along the same lines as the author's previous papers [same Žurnal 19, 86–94, 207–214, 868–875 (1949); 21, 454–461 (1951); these Rev. 10, 580; 11, 401; 12, 887]. The five-dimensional field theory is set up with the restriction that the fields be periodic with period $(2\pi\hbar/mc)$ in the fifth coordinate. It is shown that the five-dimensional field is then equivalent to a superposition of an infinite number of ordinary four-dimensional fields, one corresponding to each component of the Fourier series expansion of the total field in the fifth coordinate. The term of order Z in the Fourier series gives a four-dimensional field whose quanta are particles with rest-mass $|Z|m$ and charge Ze. Here Z may be a positive or negative integer or zero. The particles may have spin zero or one, according to the form which is chosen for the five-dimensional field Lagrangian.

F. J. Dyson (Ithaca, N. Y.).

Bergmann, Peter G., and Schiller, Ralph. Classical and quantum field theories in the Lagrangian formalism. Physical Rev. (2) 89, 4–16 (1953).

In the first part of this paper the authors examine the relation between the transformation properties of a classical field theory and its constraints, conservation laws, and generating functionals. They assume that the field theory may be derived from a variational principle whose Lagrangean (except for an additive divergence) contains only the field variables and their first derivatives. In the second part of the paper they propose a form for a quantum field theory in which field variables canonically conjugate to constraints are not observables. Their proposed field equations are derived by requiring that the Feynman-Schwinger action operator be stationary only with respect to those

variations that correspond to invariant transformations and not to all transformations. *A. H. Taub* (Urbana, Ill.).

Hurst, C. A. The enumeration of graphs in the Feynman-Dyson technique. Proc. Roy. Soc. London. Ser. A. **214**, 44–61 (1952).

In Feynman's formulation of electrodynamics [Physical Rev. (2) 76, 749–759, 769–789 (1949); these Rev. 11, 765], probability amplitudes are calculated by an expansion in powers of the fine-structure constant α. The term of order α^n is obtained as the sum of a finite number of integrals, each integral being obtained directly from the Feynman graph which represents pictorially the physical process to which the integral corresponds. The author here gives a method of calculating exactly the number of graphs appearing at a given order n. The calculation is a straightforward but rather complicated exercise in combinatorics. The number of graphs appears as a coefficient of a generating function which is a product of Hermite and Laguerre polynomials, and can be easily evaluated for any given n. The asymptotic form of the number of graphs for large n is found to be of the order $n^{\frac{1}{2}n}$. This means that unless the integrals individually tend very rapidly to zero, or unless they cancel each other very exactly, the series expansion in α will not be convergent.

The author also estimates the number of graphs which are reducible, in that they contain in their interior subgraphs of certain specified types. Such reducible graphs are easier to handle in practical calculations than irreducible ones. He finds surprisingly that the ratio of reducible to irreducible graphs of order n tends to zero for large n. This result holds under very general conditions concerning the precise way in which reducibility is defined. *F. J. Dyson.*

Hurst, C. A. An example of a divergent perturbation expansion in field theory. Proc. Cambridge Philos. Soc. **48**, 625–639 (1952).

The author investigates the simplest possible nonlinear relativistic quantum field theory. This is the theory of a single scalar field ϕ representing neutral particles, the particles interacting with each other by virtue of a cubic term $\lambda\phi^3$ in the Lagrangian of the system. This theory is a useful one since it reproduces essentially all of the physical and mathematical difficulties inherent in quantum electrodynamics and meson theory, only without the additional and nonessential complications introduced by having many field-components.

With this simple theory the author is able to estimate quantitatively the order of magnitude of the integrals which appear at the nth term of a perturbation expansion in powers of the coupling constant λ. The estimate requires a detailed study of the geometry of a $4n$-dimensional space which is too complicated to be summarized here. The result is that the nth order term of the expansion is of the order $\lambda^n n^{\frac{1}{2}n}$, and the series is therefore divergent for all λ. This result is however still incomplete for two reasons: (i) only irreducible graphs are considered; (ii) it is assumed that the system has a total energy below the threshold for creation of new particles. These limitations on the result are probably not essential; it is likely that the series diverges in all cases. See the preceding review. *F. J. Dyson* (Ithaca, N. Y.).

Cini, M. The commutation laws in the theory of quantized fields. Nuovo Cimento (9) **9**, 1025–1028 (1952).

The relation between Schwinger's theory of quantized fields [Schwinger, Physical Rev. (2) **82**, 914–927 (1951); these Rev. **13**, 520] and Peierls' rule for commutation relations [Peierls, Proc. Roy. Soc. London. Ser. A. **214**, 143–157 (1952); these Rev. **14**, 520] is discussed. Using Schwinger's theory the Peierls relations are obtained. It is shown why the Peierls rule fails in the case of arbitrary functions of field variables. *K. M. Case* (Ann Arbor, Mich.).

Kothari, L. S. On a modified definition of Riesz potential for the meson case. Proc. Phys. Soc. Sect. A. **65**, 930–933 (1952).

A modified definition of the Riesz potential for the meson field is introduced. It is shown that the new definitions is a generalization in the α-plane of the meson potential in the interaction representation. The relations satisfied by the meson potential in the interaction representation and in the Schrödinger representation are compared. (Author's summary.) *A. H. Taub* (Urbana, Ill.).

Hamilton, J. Real and virtual processes in quantum electrodynamics. Proc. Cambridge Philos. Soc. **48**, 640–651 (1952).

Quite generally in quantum mechanics, and not only in field theory, the direct calculation of the S-matrix by perturbation theory becomes troublesome when some possible intermediate states have the same energy as the initial state. In such cases it is always simpler to use not the S-matrix but the reaction matrix K which is related to S by

$$(*)\qquad S=(1-\tfrac{1}{2}iK)/(1+\tfrac{1}{2}iK).$$

A simple and general discussion of the K matrix is given by B. A. Lippmann and J. Schwinger [Physical Rev. (2) **79**, 469–480 (1950); these Rev. **12**, 570]. The paper under review is devoted to a detailed verification of equation (*), considering the explicit form of the nth term in the perturbation expansion of S in quantum electrodynamics. It is shown how the effects of virtual intermediate states, which alone are included in K, can be separated precisely from the effects of real intermediate states which appear in S.

F. J. Dyson (Ithaca, N. Y.).

Kar, S. C. Versuch einer logischen Quantendynamik des Elektrons. I. Bull. Calcutta Math. Soc. **44**, 1–21 (1952).

In his introductory section the author explains that the orthodox quantum mechanics is illogical because the principles on which it rests are separate from those of classical mechanics. He proposes that we accept quantum mechanics as logical only if there is a method of deducing logically the principles of quantum mechanics from classical mechanics and vice versa. The rest of the paper describes a new quantum mechanics which is supposed to be logical in this sense. The Hamilton-Jacobi equations describing the motion of a classical spinning electron are first written down. The quantum equations are then derived by replacing the derivatives $(\partial S/\partial x_\mu)$ in the classical Jacobi equation by the differential operators $(h\partial/\partial x_\mu)$. Here h is the usual action-constant, but the imaginary i which appears in ordinary quantum mechanics is absent. There is no wave-function and no probability interpretation of the quantum equations. It is stated that the quantum equations describe particle orbits just like the classical equations. The second half of the paper consists of algebraic manipulations of the quantum equations. The physical consequences of the theory will be discussed in later publications. *F. J. Dyson.*

Yappa, Yu. A. On a connection between theories of regularization and theories of particles with arbitrary spin. Doklady Akad. Nauk SSSR (N.S.) **86**, 51–54 (1952). (Russian)

The method of removing divergences from quantum field theory by regularization [Pauli and Villars, Rev. Modern Physics **21**, 434–444 (1949); these Rev. **11**, 301] has a well-known connection with the theory of fields obeying wave-equations of higher order than the second [Thirring, Physical Rev. (2) **79**, 703–705 (1950); these Rev. **12**, 227]. In fact, the singular Δ-function, which is the Green's function for an ordinary second-order wave-equation, becomes replaced by a less singular function when the wave-equation is of order four or higher. Also, it is well-known [Bhabha, Rev. Modern Physics **17**, 200–216 (1945); these Rev. **7**, 272] that particles of spin higher than 1 are associated with fields satisfying high-order wave-equations. The present paper is concerned with the question whether a satisfactory understanding of regularization can be achieved by supposing that the electron is just the lowest state of a particle which is capable of existing in various states with different values of spin and mass. The author discusses the arguments for and against in a very qualitative way, and does not reach a definite decision. He remarks that even the formal theory of higher-spin fields is not yet properly worked out, and until much more is known about the physical consequences of higher-spin theories, no final conclusions are possible.

F. J. Dyson (Ithaca, N. Y.).

Buneman, O. Circulation in the flow of electricity: Dirac's new variables. Proc. Roy. Soc. London. Ser. A. **215**, 346–352 (1952).

The generalized four-momentum of an electron is defined as $p_i = mv_i - (e/c)A_i$ where v_i is the four-velocity and A_i the electromagnetic four-potential. It is shown that in a theory such as Dirac's [same Proc. **212**, 330–339 (1952); these Rev. **14**, 228] the Lorentz force equation implies that the circulation $\oint p_i dr^i$ around a closed curve in four-space is a constant of the motion. This enables the author to interpret the two-dimensional surfaces, specified by the Dirac variables ξ = constant, η = constant, as "vortex ribbons". The author submits "the picture of discrete vortex ribbons of strength h (or therabouts) as a possible starting point for quantization of the new electrodynamics". [Reviewer's note: The paper has an unhappy confusion between particle and continuous fluid pictures. In terms of the latter one expects m to be a density but the analysis at the top of page 350 is valid only if $dm = 0$ for arbitrary variations. This confusion was avoided by Dirac in his lectures at the Dublin Institute for Advanced Studies, July 1952, in which the above results were given in a formulation in which e and m appeared only in the ratio m/e]. *A. J. Coleman* (Toronto, Ont.).

Shôno, Naomi, and Oda, Nobuo. Note on the non-local interaction. Progress Theoret. Physics **8**, 28–38 (1952).

Theories of non-local type in which the electromagnetic vector $A_\mu(x)$ is replaced by an average of $A_\mu(x')$ weighted by a function $\eta(x, x')$ are compared with generalized field theories such as that of Bopp [Ann. Physik (5) **42**, 573–608 (1942); these Rev. **8**, 124] in which the d'Alembertian equation $\Box A_\mu = 0$ is replaced by $f(\Box)A_\mu = 0$. It is found that though they differ in some respects, particularly as regards the free boson field, by appropriate choice of η and f equivalent theories are obtained as far as concerns finiteness of self-energy and self-charge. The extensive bibliography fails to refer to the closely related papers of Steinwedel [Z. Naturforschung **6a**, 123–133, 519–522 (1951); these Rev. **12**, 888; **13**, 411]. *A. J. Coleman* (Toronto, Ont.).

Loinger, A. Un semplice modello di due campi interagenti. Nuovo Cimento (9) **9**, 1080–1086 (1952).

Van Hove [Physica **18**, 145–159 (1952); these Rev. **14**, 118] gave an example of two fields such that the eigenvectors of the interacting fields were orthogonal to the free-field eigenvectors. However, that the orthogonality is not simply a consequence of the infinite degree of freedom is shown by the author's example: two strings vibrating in the same plane, coupled point-wise, and quantized by the usual Heisenberg-Dirac procedure. *A. J. Coleman.*

Ludwig, Günther. Wie kann die unitäre Feldtheorie Strahlungsemission, Selbstenergie und Lambshift erklären? Z. Naturforschung **7a**, 248–250 (1952).

The author has previously [same Z. **5a**, 637–641 (1950); these Rev. **12**, 783] presented a quantum electrodynamics in which a single electron does not interact with itself. However, in this theory an electron bound to an atom could not radiate and would show no Lamb shift. It is suggested that these effects are due to the existence of many other charges in the world (an "Absorber"). As justification for this view it is shown that the essential property in the conventional description of these processes is that the vacuum field satisfies the homogeneous Maxwell equations. Since this is also true of the field of the absorber, it is argued that such effects can be adequately described by the suggested theory. *K. M. Case* (Ann Arbor, Mich.).

Sakata, Shoichi, Umezawa, Hiroomi, and Kamefuchi, Susumu. On the structure of the interaction of the elementary particles. I. The renormalizability of the interactions. Progress Theoret. Physics **7**, 377–390 (1952).

The authors find the general condition that a quantum theory of interacting fields should contain only a finite number of distinct divergent processes, i.e., that the theory can be made divergence-free by a finite number of renormalizations. They find the necessary and sufficient condition to be that every coupling constant occurring in the theory should have the dimension (length)^n with $n \leqq 0$. This is also the condition which Heisenberg obtained [Solvay Ber., Kap. 3, 4 (1939)] as necessary and sufficient for a perturbation theory treatment to give reasonable results in the description of high-energy processes. If any interaction with $n > 0$ exists in nature, which Heisenberg believes to be the case, then the renormalization program and the use of perturbation theory both become unworkable and we must assume the existence of a "fundamental length" which modifies the form of the theory at high energies. On the other hand, if all existing interactions have $n \leqq 0$ then there is no reason why the theory should not be valid at all energies. Up to date the experimental evidence favors the second alternative.

F. J. Dyson (Ithaca, N. Y.).